Geometric Constructions
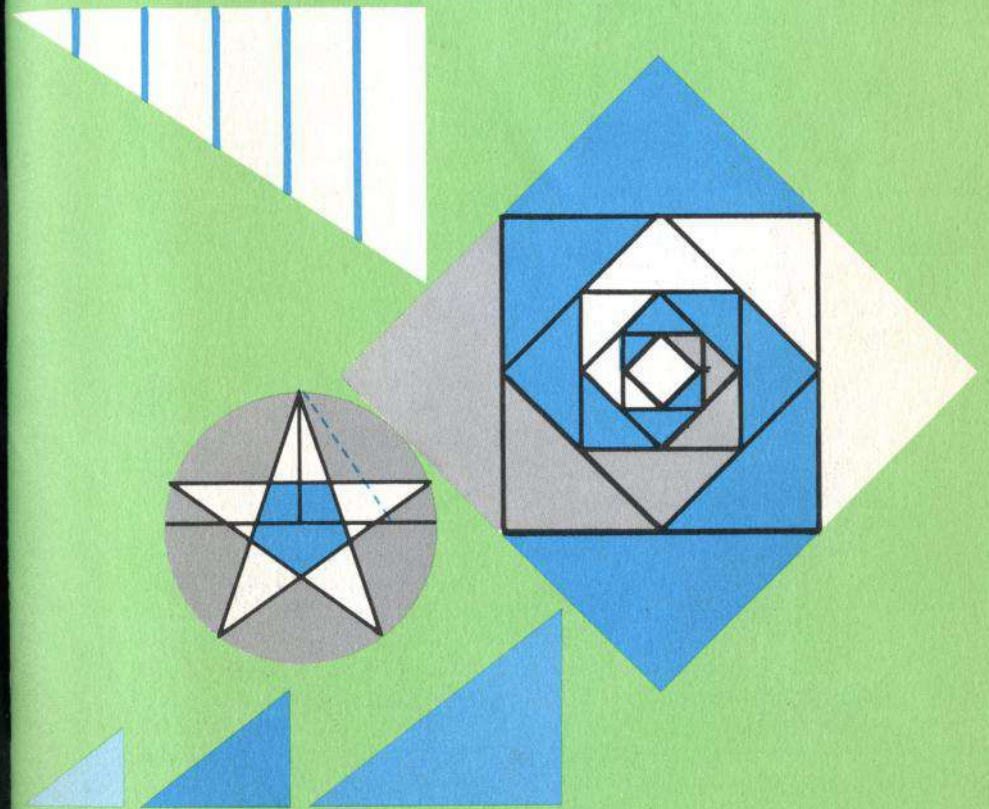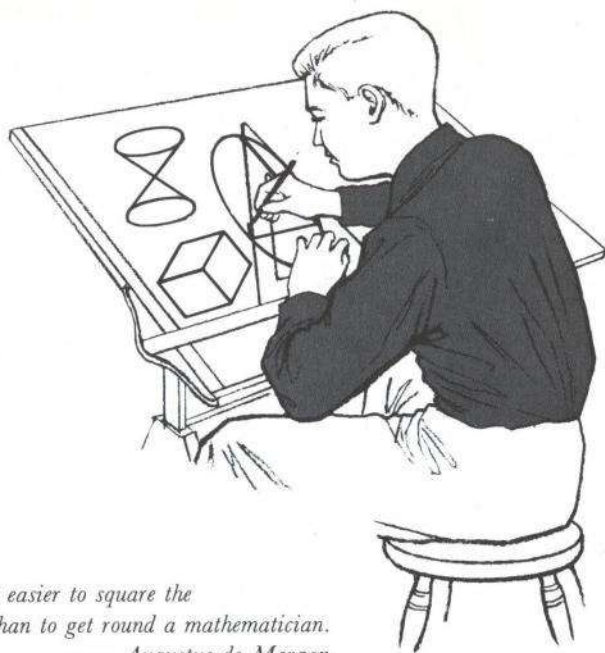
It is easier to square the
circle than to get round a mathematician.
—— Augustus de Morgan

BEFORE YOU BEGIN TO READ . . .

Reading about mathematics can be an adventure like reading a mystery story or exploring a cave. There are many surprises, puzzles, tricks, and interesting ideas in mathematics. If you will do some exploring in mathematics by yourself, you will enjoy discovering new ideas. One such topic to explore is geometric construction. This study can be both exciting and rewarding.

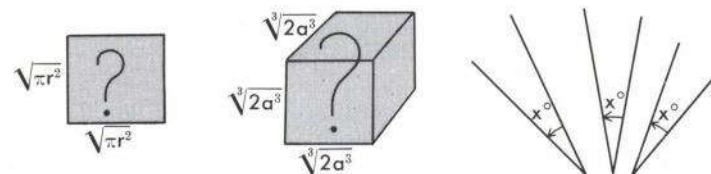You will probably have to read this booklet on geometric construction differently from the way you would read a story. To begin with, you should read it slowly. Don't be surprised if you don't quite understand every sentence or every paragraph the first time. Be patient. Get into the habit of reading mathematics with a pencil and paper within easy reach. Don't hesitate to use them. If you work the exercises, do the constructions, and complete the problems, it will be easier for you to understand what you read.

We hope that this booklet will enable you to share the pleasure others have had in exploring geometric construction.
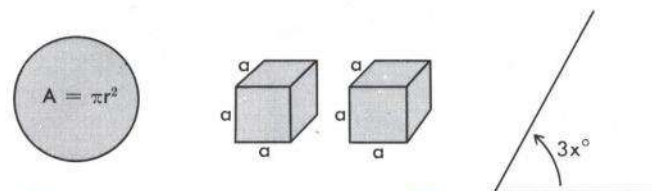
Geometric
Constructions

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The Mathematics of Geometric Construction

Circles and Lines — A Story Old and New

"Nothing's impossible!" "Where there's a will, there's a way." Do you believe this? If so, you'll find many surprises ahead as you study this booklet. There are some interesting "impossibilities" to consider as you explore the topic of geometric construction on your own.

Geometric construction is a very challenging and useful aspect of geometry. We don't know its exact origin, but it seems certain that it was studied as early as 600 years before the birth of Christ. The properties and relationships of geometric figures stirred the imagination of ancient peoples, and in our modern world this interest has continued to grow.

The ancient mathematicians found problems involving straight lines and circles of special interest. This interest is reflected in the architecture, art, and design of these early times.

How Much Is Twice as Much?

How would you construct a straight line that is twice as long as a given line segment? How could an angle that is twice the size of a given angle be constructed? How would you construct a cube that is twice the volume of a known cube? The answers to the first two questions are quite easy, but let's give the third question a second thought.
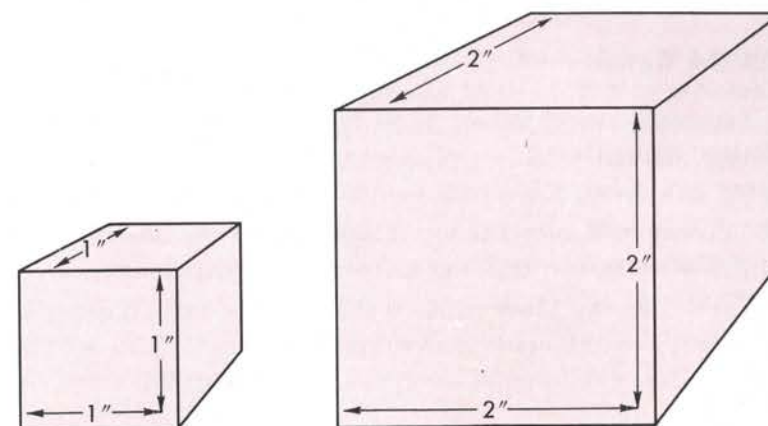
Figure 1

Suppose we had a cube with a volume of 1 cubic inch. Our problem is to construct a cube that has twice the volume of the given cube, which has a volume of 1 cubic inch. Remember that the volume of a cube is found by the formula

$$V = s \times s \times s \quad \text{or} \quad V = s^3.$$

In the case of a cube with sides of 1 inch,

$$V = 1'' \times 1'' \times 1''$$
$$= 1 \text{ cu. in.}$$

What happens to the volume of the given cube if we double the length of its sides (Figure 1)? The formula then applies as follows:

$$V = (2'')^3$$
$$= 2'' \times 2'' \times 2''$$
$$\bullet = 8 \text{ cu. in.}$$

By doubling the sides of the known cube with sides of 1 inch, the volume becomes eight times greater!

How much must we increase the sides of the 1-inch cube to double its volume? This is known as the problem of *duplicating the cube*. It is one of the famous construction problems in mathematics. We will consider in more detail the "duplicating the cube" problem and other such famous construction problems later in this booklet. Some of these fascinating problems have histories dating back hundreds of years. You may be surprised to learn that many of these problems are impossible to solve. First, we have to consider the nature of geometric construction, how to approach geometric construction problems, something of the criteria for constructibility, and some basic constructions which are important in almost all problems encountered.

Rules of the Game

In geometric construction, as in football, basketball, baseball, and other games, certain rules govern our actions. If we violate a rule, our actions, or scores, are nullified.

Geometric construction is confined to the use of the compass and an unmarked straightedge.

It is said that Plato first outlined the rule that limited geometric construction to a compass and straightedge. Plato also set forth the following assumptions concerning the uses of these two instruments:

1. A straight line of any length could be constructed through any two known points.
2. A circle could be constructed with a known point as center and a radius determined by a second known point.

In construction, measuring with a ruler, using a protractor, or using even one mark on the straightedge are all violations of the game. This idea will be brought out in many instances as you explore work in this booklet.

For example, although the trisection of any angle is impossible under the definition of construction as set forth by the ancient mathematicians, such an angle can be trisected by compass and a marked straightedge. We will explore this idea in more detail later in this booklet.

There is a definite difference in mathematics between construction and *drawing*. You could draw geometric figures by use of many various devices. The draftsman uses the T square, parallel rulers, drawing triangles, rulers, and curve instruments to make drawings. In geometric construction, you can use only the un-marked straightedge and the compass. **Measurement by ruler, protractor, or other devices is prohibited.**

It is important for us to consider the *condition* of the tools used in construction. A compass in good condition, a sharpened pencil, and a proper straightedge are essential for geometric construction.

A Well-constructed Approach to Geometric Construction

"Practice makes perfect!" This well-known saying applies well to work in geometric construction. You will find that practice and experience with problems in geometric construction are necessary to the development of skill and knowledge in solving problems.

It is difficult to set forth specific steps for you to use in attacking problems in geometric construction. You will find that there are many problems to consider. Like batters in baseball, each person usually develops a successful style all his own. However, there are certain guidelines which are reasonable and helpful as you approach such problems. General guidelines suggested in approaching problems in geometric construction are as follows:

1. Take ample time to analyze the conditions of the problem and its requirements.
2. Draw a sketch of the completed construction. This sketch will help you to visualize the required result. It will also help you to discover certain relationships to apply in solving the problem.
3. In an analysis of the construction as sketched, study the given conditions, that is, existing relationships regarding the completed construction as originally stated. Feel free to use auxiliary lines and circles which might be of value.
4. Continue your analysis until you find a possible or obvious solution.
5. Complete the construction and make certain that your work satisfies all the requirements of the stated problem.
6. Review the construction steps in the solution of the problem. Note the significant relationships which led to the solution.

It Can't Be Done! Or Can It?

A straightedge can be used to construct straight line segments. A compass constructs circles or curved lines which are parts of circles. You can see that constructions with straightedge and compass involve intersections of straight lines, circles, or straight lines and circles.
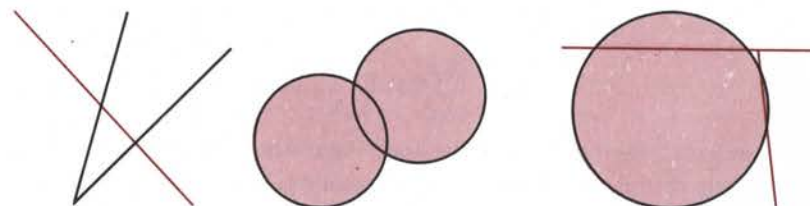

Figure 2

We won't attempt to explore completely the mathematics of *constructibility*, but let's consider several criteria that are important to our work. Don't be too concerned if the mathematical computations seem somewhat difficult. Our major purpose is to have you gain a general concept of the ideas rather than to teach you the specific mathematics presented at this point.

Let's consider the case of two straight lines which intersect. A well-known equation in algebra and analytical geometry is $Ax + By = C$. Do you recognize it? This is the general equation for a straight line.

Suppose we had two straight lines represented by the general equations

$$A_1 x + B_1 y = C_1 \quad \text{and}$$
$$A_2 x + B_2 y = C_2.$$

We can find the point of intersection by use of some basic algebra. We find this point by solving the two linear equations.

Solution:
$$A_1 x + B_1 y = C_1 \quad \text{(Line 1)}$$
$$A_2 x + B_2 y = C_2 \quad \text{(Line 2)}$$

Step 1—Multiplying Line 1 by B_2 and multiplying Line 2 by B_1 we obtain the following:

$$B_2 A_1 x + B_2 B_1 y = B_2 C_1 \quad \text{(Line 1')}$$
$$B_1 A_2 x + B_1 B_2 y = B_1 C_2 \quad \text{(Line 2')}$$

Step 2—Subtracting Line 2' from Line 1' gives

$$B_2 A_1 x + B_2 B_1 y = B_2 C_1$$
$$- (B_1 A_2 x + B_1 B_2 y = B_1 C_2)$$
$$\overline{B_2 A_1 x - B_1 A_2 x = B_2 C_1 - B_1 C_2}$$

Step 3—By factoring the result of step 2, we obtain

$$x(B_2 A_1 - B_1 A_2) = B_2 C_1 - B_1 C_2$$

Step 4—Dividing both members of the equation by $B_2 A_1 - B_1 A_2$

$$x = \frac{B_2 C_1 - B_1 C_2}{B_2 A_1 - B_1 A_2}$$

Similarly, we find that

$$y = \frac{A_2 C_1 - A_1 C_2}{A_2 B_1 - A_1 B_2}$$

We say that the solutions for x and y represent the *coordinates* of the intersection of the two lines. As you examine these results, note that these coordinates are made up of combinations of the numerical coefficients of the original line equations.

To obtain x, C_1 is multiplied by B_2 and C_2 by B_1. Then the two are subtracted to find the numerator. To find the denominator, A_1 is multiplied by B_2 and A_2 multiplied by B_1; then we subtract. We find y in a similar manner. The numerator over the denominator represents a division operation. Our solution is found by the use of the arithmetic operations of multiplication, subtraction, and division.

You could show in a similar way that the intersection of a line and a circle and the intersection of two circles involve the arithmetic operations of second-degree equations, or equations in the form $x^2 + y^2 = C^2$. We add only the extraction of square roots to the four operations of addition, subtraction, multiplication, and division when circle equations are introduced.

We can now define the criterion for constructibility which determines the solutions to problems in geometric construction.

Here is the key! Construction by straightedge and compass, *under the rules of the game*, is possible only when our construction involves a finite number of the arithmetic operations of addition, subtraction, multiplication, and division and the extraction of real square roots.

This criterion is vitally important to our exploration of geometric construction. It served as the mathematical key to unlock the mysteries of many problems which puzzled mathematicians for hundreds of years. This also served as the basis which led to the mathematical proofs for the impossibility of duplicating the cube, trisecting any general angle, or squaring the circle. We'll take a closer look at this idea later in our study.

EXERCISE SET 1

The Nature of Geometric Construction

1. Describe in your own words what is meant by "criterion for constructibility." How is this important in exploring problems in geometric construction?

2. Construct a circle with your compass. Keeping the same compass opening, place the compass point on any point on the circumference of the circle and swing an arc intersecting the circle. On this point of intersection, swing another arc. Continue until six points are marked. Connect these points with straight lines. What figure is formed?

3. Using the six points on the circle in Exercise 2, use straight lines to connect alternate points. What figure results? How many triangles can you count in the final figure?
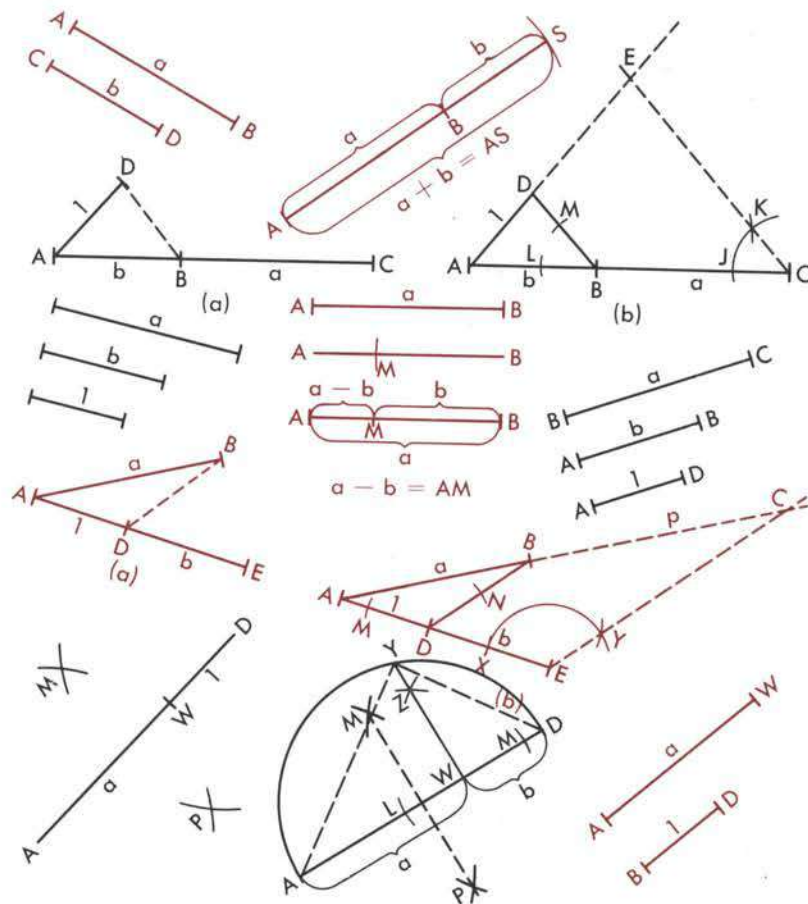
Arithmetical Computations by Construction

Five Lines Make a Point

An important objective for us is to determine whether or not the arithmetic operations of addition, subtraction, multiplication, and division and the extraction of square roots can be constructed.

That is, can we construct $a + b$, $a - b$, $\frac{a}{b}$, $a \times b$, and $\sqrt{a}$?

Let's explore each of these possibilities separately. Suppose we begin by considering the construction of $a + b$.

I Construct line $a + b$, given line a and line b.
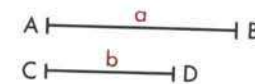
Figure 3

Step 1 — Place the compass point at C on line b and open the compass equal to the length of b.

Step 2 — Keeping the compass opening of step 1, place the point of the compass at B on line a. Swing an arc to the right of B. Continue line AB until it intersects the arc at point S. Line AS is our required construction.
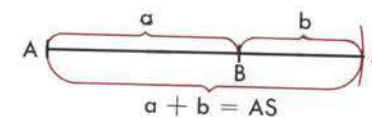

$$a + b = AS$$

Figure 4

II Construct line $a - b$, given line a and line b; $a > b$.
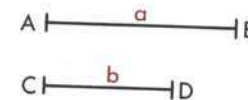

Figure 5

Step 1 — Place the compass point at C on line b and open it equal to the length of line b.

Step 2 — Using the compass opening of step 1, place the compass point on point B of line a. Swing an arc to the left of B, intersecting line a at M.

Line AM is required construction $a - b$.
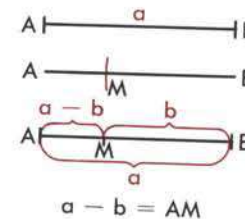

$$a - b = AM$$

Figure 6

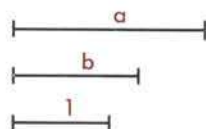### III Construct a line $a \cdot b$, given line a, line b, and the unit line.

Figure 7

Note: An important theorem of geometry states that a line parallel to one side of a triangle and intersecting the other two sides divides the sides proportionally.

Step 1 — Construct the unit length represented by 1, and at its end point add line b. Using any convenient angle, lay off line a as shown in Figure 8a. Connect points B and D.
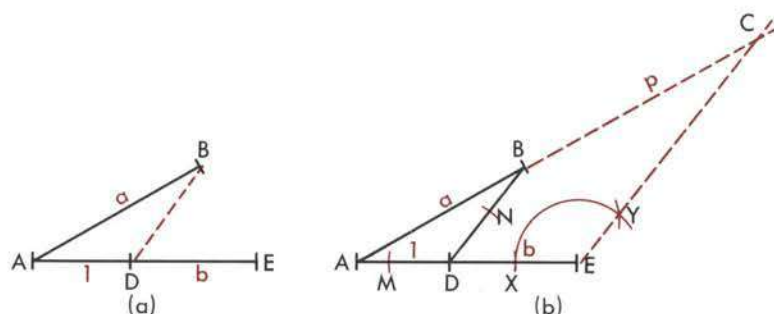

Figure 8

Step 2 — Place your compass point at D and swing an arc which intersects AD and BD at points M and N respectively. Keep the same compass opening, and swing an arc at point E which intersects line DE at X. Place one compass point at M and the other compass point at N. Using this compass opening, swing an arc from point X to intersect the previous arc at point Y, as in Figure 8b.

Step 3 — Using points E and Y, construct a straight line intersecting line AB extended at C.

Step 4 — Since angles ADB and DEC are equal by construction, BD and CE are parallel. Thus, by the theorem pointed out to you on this page,

$$\frac{p}{a} = \frac{b}{1}.$$

Step 5 — Since in any proportion the product of the means equals the product of the extremes,

$$p = a \cdot b.$$

Thus, line p represents our required construction.
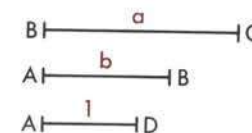
IV Construct $\frac{a}{b}$, given lines a and b.

Figure 9

Step 1 — Construct a line $b + a$; using any convenient angle, construct the unit line as shown in Figure 10a. At point B
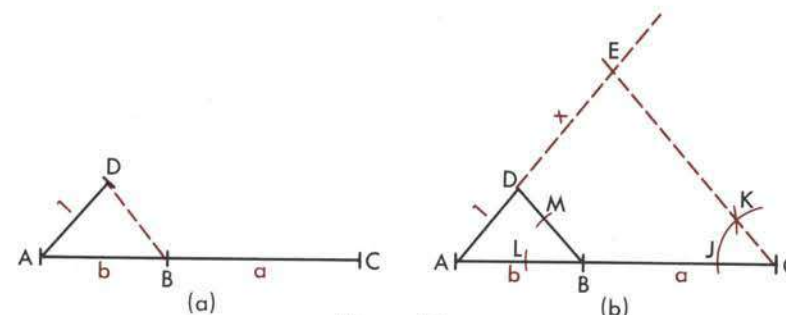

Figure 10

on line AC, swing an arc intersecting AB and DB at L and M. Keeping the same compass opening, swing an arc from point C intersecting line AC at point J. Use the two compass points to "measure" the angle at points L and M. Using this compass opening, swing an arc from J to intersect the previous arc at K.

Step 2 — Draw a straight line from point C through K, intersecting AD extended at E.

Step 3 — By the theorem used previously, $\frac{x}{1} = \frac{a}{b}$.

Step 4 — By multiplying both sides of $\frac{x}{1} = \frac{a}{b}$ by 1, we obtain the desired result, $x = \frac{a}{b}$. Thus, line x is our required construction.

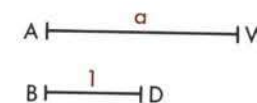### V Construct the square root of a line, given the line and a unit length.

Figure 11

Step 1 — Construct a line equal to $a + 1$, where 1 represents the unit line.

Step 2 — From points A and D, swing arcs of equal radius above and below AD so that the arcs intersect at M and P.
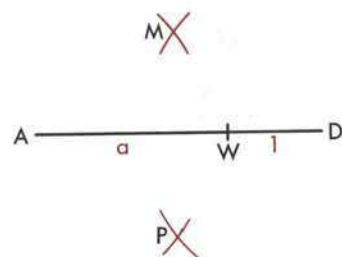

Figure 12

Step 3 — Using a straightedge, connect points M and P. Mark point O, the intersection of MP and AD. Using a compass opening equal to OD or OA, complete the semicircle as shown in Figure 13.
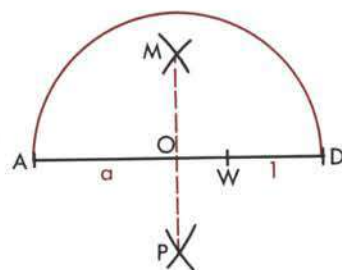

Figure 13

Step 4 — Place the compass at W, and with a convenient compass opening swing arcs to intersect AD at L and X.

Step 5 — Place the compass point at L, and with a wider compass opening swing an arc above AD. Repeat the process, using the point X. Where these arcs meet at Z, construct perpendicular line ZW which intersects the arc of the semicircle at Y. Connect YD and AY. What kind of an angle is angle AYD? You are correct if your answer is 90°, or a right angle. Thus, triangle AYD is a right triangle.

Note: Another important geometric theorem states that an altitude drawn to the hypotenuse of a right triangle is a mean proportional between the segments of the hypotenuse. That is, in Figure 14, $\dfrac{AW}{YW} = \dfrac{YW}{WD}$.
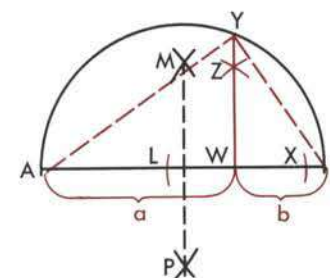

Figure 14

Step 6 — By multiplying both sides of the proportion $\dfrac{AW}{YW} = \dfrac{YW}{WD}$ by YW and then by WD, we get the result of

$$(YW)^2 = (AW)(WD).$$
$$\text{Since } WD = 1,$$
$$(YW)^2 = AW.$$
$$\text{Therefore, } \sqrt{(YW)^2} = \sqrt{AW}$$
$$\text{and } YW = \sqrt{AW}.$$

The perpendicular line YW represents our required construction.

A Major Point Revisited

You'll recall our earlier work and findings regarding the key to geometric construction. We said that construction by straightedge and compass was possible only when our required construction involved a finite number of the rational operations of addition,

subtraction, multiplication, and division and the extraction of real square roots.

In our study in the previous section, we demonstrated that it is possible to construct each of the rational operations as well as to construct the square root of a given line segment. We say that if the construction involves no irrational operations other than that involved in extracting real square roots, the construction is possible.

A New Look at a Radical Construction

Earlier you explored the construction of the square root of a given line segment. Let's consider another method for constructing a radical, or square root.

This method is divided into two parts. Part One describes the arithmetic necessary to prepare for the construction. Part Two describes the method of construction and the use of the information obtained in Part One.

Part One

Suppose you were asked to construct the square root of 15. What is the largest whole number whose square is 15 or less? Our answer must be 3, since $3^2 = 9$ and the next larger number is 4 and $4^2 = 16$, which is larger than 15. Let's use 3 as one of the sides of a right triangle, with a hypotenuse of $\sqrt{15}$ (Figure 15).
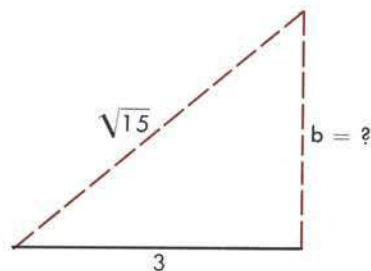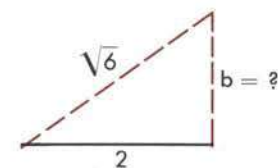

Figure 15

By the famous Pythagorean theorem, $(\sqrt{15})^2 - 3^2 = b^2$. By arithmetic, $15 - 9 = b^2$. Thus, $b^2 = 6$, and $b = \sqrt{6}$.

Next, let's use $\sqrt{6}$ as the hypotenuse of a second right triangle and repeat the same process. The largest whole number whose square is less than 6 is 2. Our right triangle will have a hypotenuse of $\sqrt{6}$ and one side of 2.

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$$b^2 = (\sqrt{6})^2 - 2^2 = 6 - 4 = 2$$
$$b = \sqrt{2}$$

Figure 16

What are the sides of any right triangle with a hypotenuse of $\sqrt{2}$? You are correct if your answers are 1 for both sides.
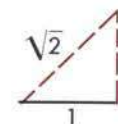

Figure 17

Part Two

Once we have performed our arithmetic, we have a guide to the procedure necessary to construct a line equal to $\sqrt{15}$.

First, we select an arbitrary unit line. Then we construct $\sqrt{2}$. Using the length $\sqrt{2}$, we construct the line $\sqrt{6}$. Finally, we use the length $\sqrt{6}$ to construct the length $\sqrt{15}$ as in Figure 18.
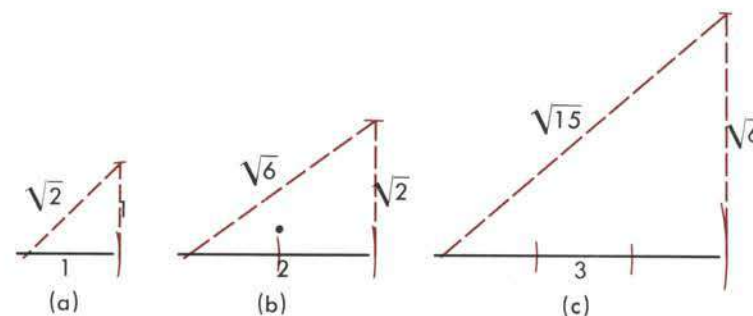

(a)   (b)   (c)

Figure 18

Let's consider another example. Suppose we construct the radical $\sqrt{21}$. The largest integer whose square is less than 21 is 4.

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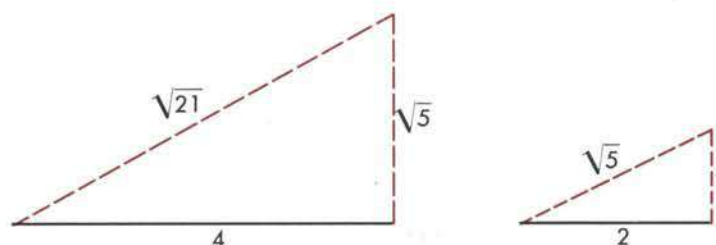**Figure 19**

Our required construction can be completed by beginning with the right triangle with legs of 1 unit and 2 units.
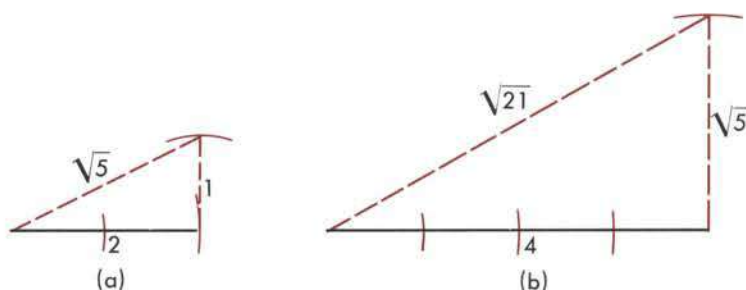

Figure 20

The hypotenuse in Figure 20*b* represents our required construction.

Arithmetical Constructions and the Basic Laws of Mathematics

How does a change in the order of events affect the result? For example, does it make a difference if we add $3 + 5$ or switch the order of the addends and add $5 + 3$? In this case, we see the change in order does not alter the sum, which is 8. In the case of $10 \div 2$, changing the problem to $2 \div 10$ makes a considerable difference.

Let's explore this idea in regard to arithmetical constructions. You'll recall our earlier work when the line segment $a + b$ was constructed. When we added a given line *a* to a given line *b*, a new segment $a + b$ resulted. Whenever two or more lines are

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added by construction, a new segment representing their sum is the result. We say that a line plus a line is another line.

Does reversing the order of adding line segments change the result? That is, for any given line segments *a* and *b*, does $a + b = b + a$? You can find the answer by studying Figure 21.
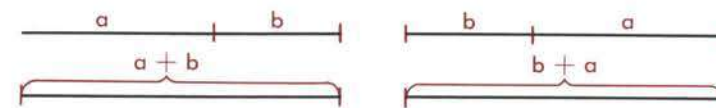

Figure 21

We see that the sums $a + b$ and $b + a$ are equal. In a mathematical set of elements, when $a + b = b + a$ for all possibilities, we say the system is *commutative* for addition.

Since for line segments *a* and *b*, $a + b = b + a$, we can say that the commutative law holds true for this example.

In the system of counting numbers, $5 + (3 + 4) = (5 + 3) + 4$, since both quantities equal 12. Similarly, $10 + (2 + 1) = (10 + 2) + 1$. We can associate the 2 with the 1 and add 10 or associate the 2 with the 10 and add 1 without changing our answer. We say that the example is *associative* for addition.

Let's explore the associative law as it applies to adding line segments by construction. Suppose we are given three line segments *a*, *b*, and *c*. Does $a + (b + c) = (a + b) + c$?
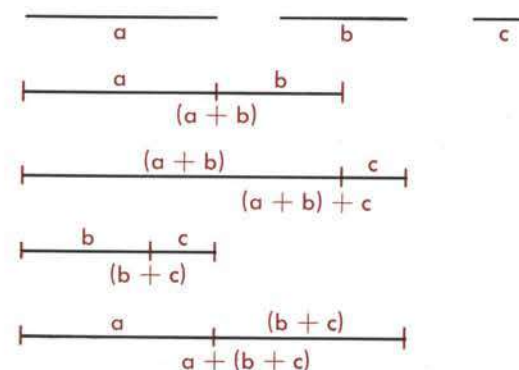
Check the constructions in Figure 22.

Figure 22

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The final results in the two constructions are equal; that is, $a + (b + c) = (a + b) + c$. We know the associative law holds true for this example.

If we are given segments a and b, does $a \cdot b = b \cdot a$? You can check our earlier work in Figure 8b for the product $a \cdot b$. Figure 23 shows the segment p_1 as the product $b \cdot a$, where segments a and b are equal to those used in Figure 8. The angle, also, must be the same for this to be true. Does segment p in Figure 8 equal segment p_1 in Figure 23? We can conclude that the commutative law for multiplication holds true for this example.
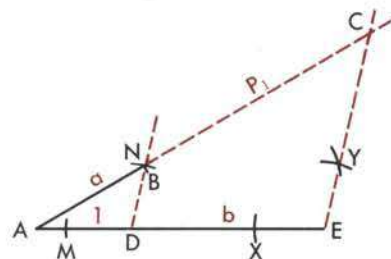

Figure 23

Another fundamental property in mathematics is called the *distributive principle*. In the system of counting numbers we say that $5(3 + 4) = 5 \cdot 3 + 5 \cdot 4$. If we add $3 + 4$, our sum is 7; $5 \times 7 = 35$. The right member of the equation also equals 35.

Let's apply the distributive principle to the construction of line segments. Given line segments a, b, and c, does $a(b + c) = ab + ac$? Let's construct the first quantity, $a(b + c)$ (Figure 24).
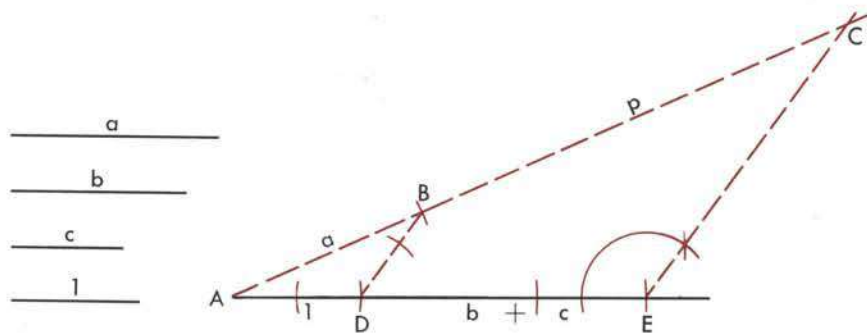

Figure 24

In Figure 24, segment p represents the construction $a(b + c)$. Our next goal is to construct the segments ab and ac and find their sum. Lay off segment a at the same angle used to construct $a(b + c)$ in constructing ab and ac.
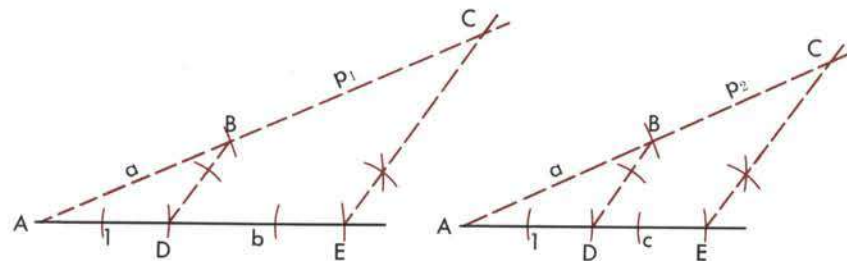

Figure 25

By construction, add segments $p_1(a \cdot b)$, and $p_2(a \cdot c)$, and compare their sum to segment p or $a(b + c)$ in Figure 24. What do you find? Since these segments are equal, we see that $a(b + c) = ab + ac$ and that the distributive principle holds true.

EXERCISE SET 2

Arithmetical Constructions

1. Given line segment a, construct the following segments. (Use the unit line where necessary.)

a. $2a$ d. $a - a$

b. $\sqrt{a}$ e. $\dfrac{a}{a}$

c. $a + a$ f. $a \cdot a$

2. Construct the following radicals.

a. $\sqrt{19}$ d. $\sqrt{26}$

b. $\sqrt{6}$ e. $\sqrt{92}$

c. $\sqrt{18}$ f. $\sqrt{87}$

3. Given line segments a, b, and c, show the following are true.

a. $a + b = b + a$
b. $b + c = c + b$
c. $a + (b + c) = (a + b) + c$
d. $a \cdot c = c \cdot a$
e. $a(b + c) = a \cdot b + a \cdot c$

4. What basic mathematical principle is illustrated in each example in Exercise 3 above?
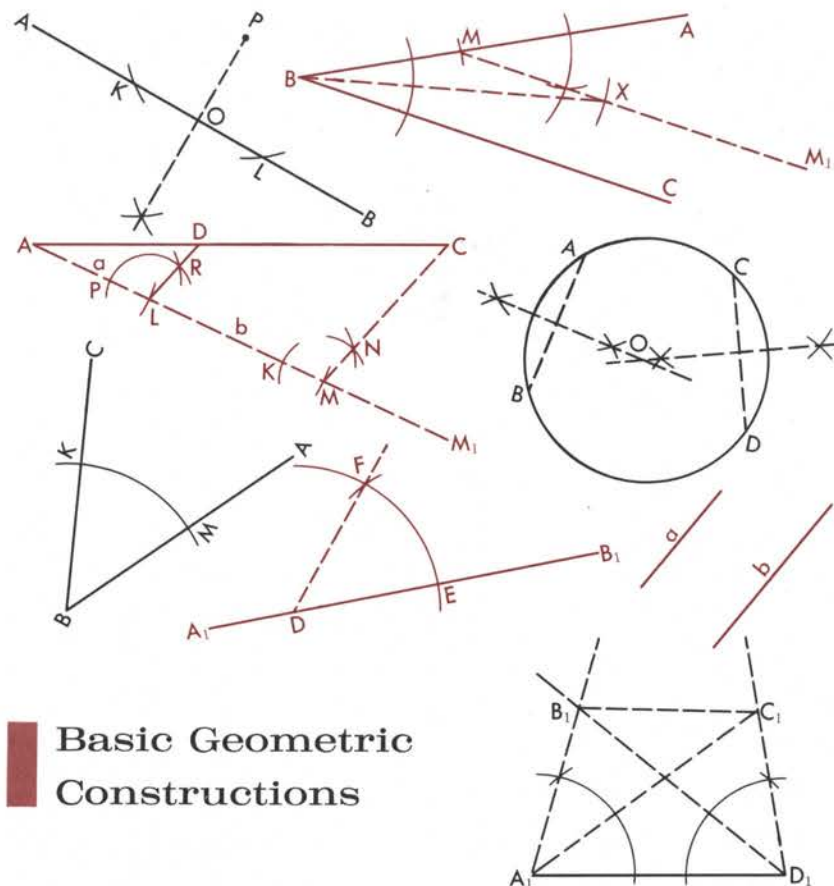
Basic Geometric Constructions

Constructing Strong Foundations

The athlete must learn the basic fundamentals of a sport before he attempts to participate in an important game. There are certain basic constructions which are used in complex geometric construction problems.

Some of these basic constructions have been studied already. For example, in several earlier problems we constructed parallel lines and copied angles and given line segments. We'll explore many other important and challenging fundamental constructions in this chapter. It is important that you be able to complete these basic constructions easily. You'll find this knowledge valuable when you explore other constructions later in this booklet. Follow the steps presented for each construction carefully.
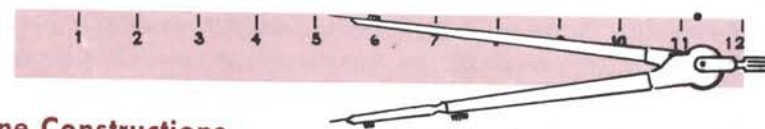
Line Constructions

Let's consider some basic constructions with lines. Suppose we begin by considering the bisection of a given line segment.

1. Bisecting a line

To bisect any given line, place your compass at an end point of the segment. Using a radius that is greater than one-half the length of the line, swing an arc that extends well above and well below the line segment. Now repeat the process, using the other end point of the given line segment and the same compass opening. The arcs should meet in two points. Using your straightedge, line up the points of intersection and construct a line which intersects the given line.
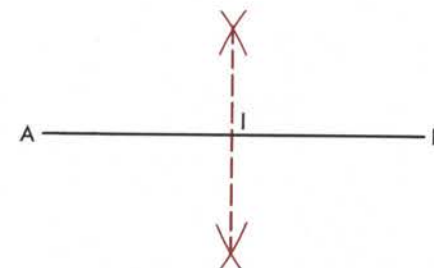

Figure 26

The point of intersection, point I in Figure 26, represents the middle, or midpoint, of the given line. The point of intersection divides the line into two equal segments. We say the line is *bisected*.

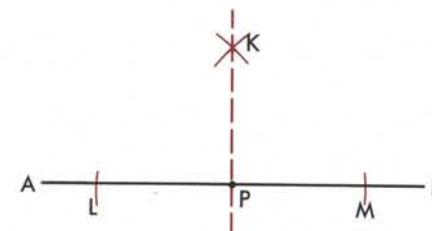#### 2. Constructing a perpendicular to a given point P on a given line l

Figure 27

You'll find that knowing how to construct perpendicular lines is very important. A perpendicular line to AB at point P is a line which meets line AB at right angles.

Let's begin by placing the compass at point P and swinging arcs which intersect AB at L and M. Using a wider compass opening, swing arcs from L and M to intersect at K. Using points K and P, construct a line which intersects AB. The line KP is our required perpendicular. We call angles BPK and APK right angles; both have 90°.

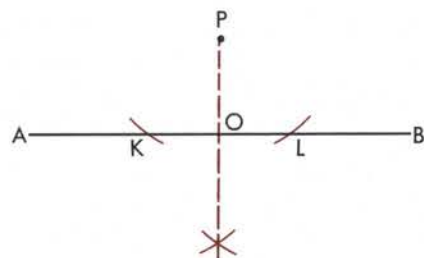### 3. Constructing a perpendicular to a given line from a point not on the line

Figure 28

From point P, swing an arc intersecting AB at K and L. From points K and L, swing arcs above or below AB so that the arcs intersect. Use the point of intersection and point P to construct the required perpendicular intersecting AB at O. What kind of angles are LOP and KOP?

4. Copying an angle

One of the most frequent constructions used in this type of work is the copying of a given angle.
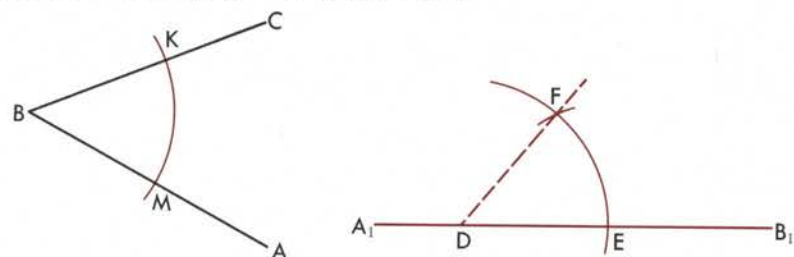

Figure 29

In Figure 29, we are given angle ABC. Draw any line A_1B_1. Select any point D on line A_1B_1. Using any convenient compass

opening, place one point of your compass at the vertex of the angle to be copied (in this case, point B). Swing an arc that will intersect both sides of angle ABC. Side BC will be intersected at some point K, and BA will be intersected at some point M. Using the same compass opening, and with one point of your compass at D, swing an arc to intersect A_1B_1 at point E. Using the length KM as your compass opening, place one point of your compass at E and swing an arc that will intersect your previous arc at F. Draw the line determined by DF. The angle EDF is equivalent to angle ABC.

5. Constructing two lines which are parallel

There are several methods of constructing parallel lines. You may sometime be asked to construct a line which passes through a given point and is parallel to a given line.

Suppose we consider the construction of a line parallel to a given line. Let's begin the construction by drawing any line AB and marking any two points X and Y on the line. Construct perpendicular lines at both points X and Y.

Figure 30

Place the compass at point X and swing an arc intersecting the perpendicular at point M. Using the same compass opening, place the compass point at point Y and repeat this process, locating point N. Construct a line through points M and N; this line is our required parallel.

Now let us consider the problem of constructing a line that passes through a given point and that is parallel to a given line. Draw any line AB and mark some point P that is not on the line (see Figure 31). Using a line drawn through P, intersect AB at some point X. Place one point of your compass at X, and using any compass opening, swing an arc that intersects AB at Y and XP at Z. Using the same compass opening, with one point of your compass at P swing an arc to the same side of XP as the angle PXB.
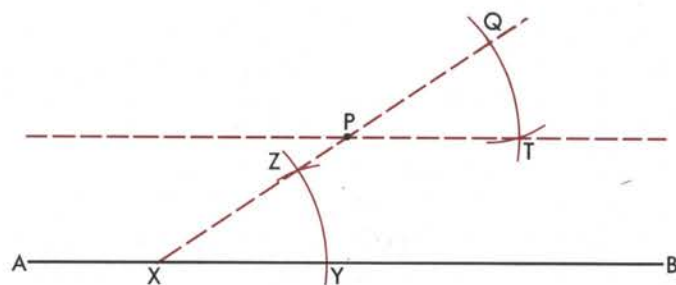
Figure 31

This arc should intersect XP at Q (P is between X and Q). Now using the length YZ as a compass opening, with one point of your compass at Q swing an arc that intersects the previous arc at some point T. Draw the line that is determined by the points P and T. PT is the required line parallel to AB which passes through P.

6. Dividing a given line into any number of equal parts

You'll find that many constructions require you to divide a line into three or more equal parts. Let's consider dividing a line by construction into five equal parts.

At the end point of given line TR, construct a second line TL at any convenient angle, as in Figure 32. Using any convenient compass opening, swing an arc from the vertex of the angle formed by the intersection of TR and TL.
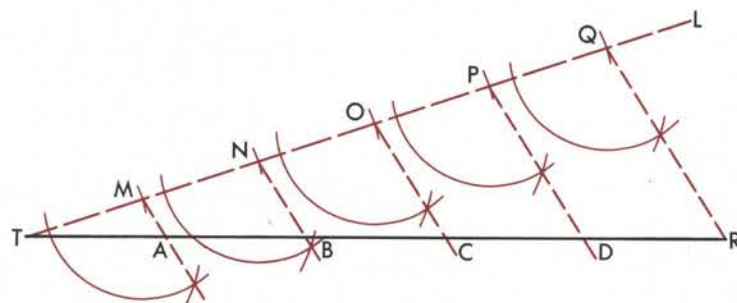

Figure 32

From point M, using the same compass opening, swing a second arc on TL intersecting at N. Continue this process until five line segments are determined on TL. Connect points Q and R. By copying angle PQR at points P, O, N, and M, construct four lines parallel to QR intersecting line TR at points A, B, C, and D. A, B, C, and D divide line TR into five equal parts as required.

7. Dividing a given line segment into parts which are proportional to two given line segments

Many constructions will require you to divide a line segment proportionally. In this specific example, you must divide a given segment AC into parts which are proportional to given segments a and b.

Let's begin our construction by marking a line AM_1 at any convenient angle from the end point A of the given line segment AC (Figure 33). Next, from point A, using a compass opening equal in length to the given segment a, swing an arc that will intersect AM_1 at L. From L, swing an arc that is equal in length to the given segment b. This will intersect line AM_1 at M. Draw the line segment CM.
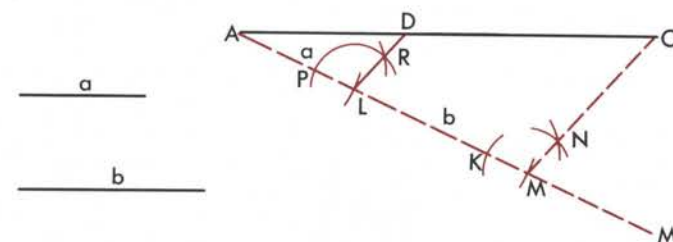

Figure 33

Place one point of your compass at M, and using any desired compass opening swing an arc that intersects AM at K and CM at N. Using the same compass opening and one point of your compass at L, swing an arc that will intersect AM at P (P should be between points A and L). Using the length KN for your compass opening, with one point of your compass at P swing an arc that will intersect the previous arc at some point R. (R should be on the same side of AM as the line AC.) Draw the line determined by the points L and R which will intersect AC at some point D. The point D divides segment AC into the required proportion. We say that AD is to DC as a is to b. Sometimes we write this proportion as

$$\frac{AD}{DC} = \frac{a}{b}.$$

8. Constructing a line which is a fourth proportional to three given line segments

In any proportion such as $\frac{1}{2} = \frac{4}{8}$, we call the 8 a *fourth proportional*. Similarly, in the proportion $\frac{m}{n} = \frac{x}{y}$, y is the fourth

proportional. We refer to the 2 and 4 and the n and x as the *means* and the 1 and 8 and the m and y as the *extremes* of the proportion.

In construction 8, you are asked to construct a line which is a fourth proportional to the three lines a, b, and c.

Let's begin our construction by constructing the line $a + b$. In Figure 34, the point M divides AD into two segments, of which $AM = a$ and $MD = b$. Through point A, draw a line at any convenient angle to AD. Place one point of your compass at point A, and with a compass opening the length of line c swing an arc that intersects AB at point N. Connect points M and N.
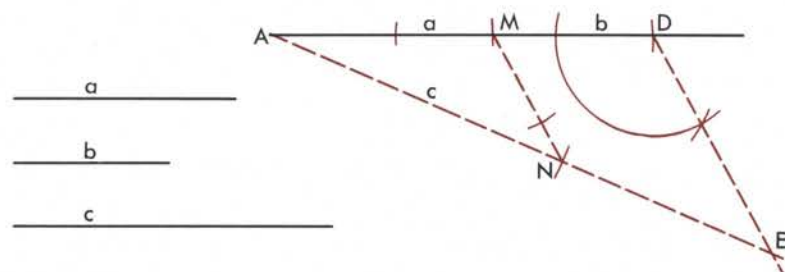

Figure 34

At point D, copy the angle AMN and extend its side to intersect AB at E. Lines NM and ED are parallel. Thus, $\dfrac{AM}{MD} = \dfrac{AN}{NE}$. If we use small letters to name the line segments, our proportion becomes $\dfrac{a}{b} = \dfrac{c}{x}$, and segment x represents our required fourth proportional.

9. Constructing a mean proportional to two given line segments

In the proportion, $\dfrac{a}{c} = \dfrac{d}{b}$, it is possible for c and d to be equal. In such a case, the proportion $\dfrac{a}{c} = \dfrac{d}{b}$ is equivalent to $\dfrac{a}{x} = \dfrac{x}{b}$. We say that x is a *mean* proportional between a and b. This problem asks you to construct a mean proportional between two line segments which are known.

Let's begin the construction by drawing a working line l. On this working line, mark off the lengths of segments a and b with the compass.
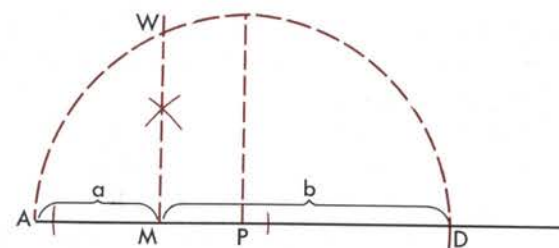
Figure 35

Bisect line AD, which is equivalent to $a + b$. Using the midpoint P of AD as center and PA as radius, construct a semicircle $\widehat{AD}$. At point M, the point which divides segments a and b, construct a perpendicular which intersects the semicircle at W. Line MW is our required mean proportional.

$$\frac{AM}{MW} = \frac{MW}{MD} \quad \text{or} \quad \frac{a}{x} = \frac{x}{b}$$

You learned earlier that the altitude drawn to the hypotenuse of a right triangle is a mean proportional between the two segments of the hypotenuse. In Figure 35, what figure is formed by connecting points A and W, then W and D?

Working the Angles — Extending Basic Constructions

Earlier you constructed the bisector of a line. Suppose we begin our exploration in angle constructions by considering the bisecting of an angle.

1. Constructing the bisector of a given angle

There are several interesting methods which can be used to bisect any given angle. Let's consider two possibilities.

a. From the vertex A of the given angle swing an arc which intersects each side of the angle at L and K. From points K
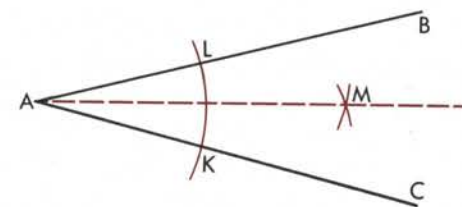

Figure 36

and L, swing arcs using the same compass opening so that they intersect at M. Points A and M determine the line AM and the required angle bisector.
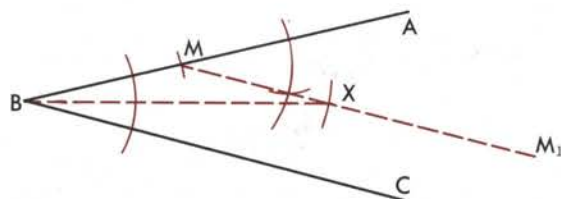
b. Angle ABC is to be bisected.

Figure 37

From point B, swing an arc of any convenient length to intersect BA at M. At point M, construct MM_1 parallel to BC. From M, swing an arc of length BM which intersects MM_1 at X. Draw BX. BX is the required bisector of angle ABC.

An Angle Bisector Device

This angle bisector device involves the same principles for bisecting an angle as the usual ruler and compass construction. The device consists of four bars which may be cut from cardboard, wood, metal, or plastic (Figure 38). If cardboard is used, the device can be assembled with brass paper fasteners. If the heavier materials are used, $\frac{1}{8}$-inch bolts and washers should be used. Suggested dimensions for the device are as follows: OR and OT each 20 inches long, AF and BF each 9 inches long, and OS 24 inches long. As the diagram illustrates, the bar OS must have a slot in it. In order that the device may be used to bisect a large range of angles, the slot in OS should be made as long as the strength of the material will permit.

To assemble the device, first locate A and B on bars OR and OT respectively so that $OA = OB$. Drill holes at O, A, B, and F on the various bars as needed, insert bolts and washers as needed, and the construction is complete. Since triangles BOF and AOF are congruent for all positions of F along OS, the device can be used to bisect any angle AOB within the limits allowed by its mechanical features.

This device can be used to demonstrate the principles involved

in constructing the perpendicular bisector of a line segment. It can also be used to find the center of a circular object.
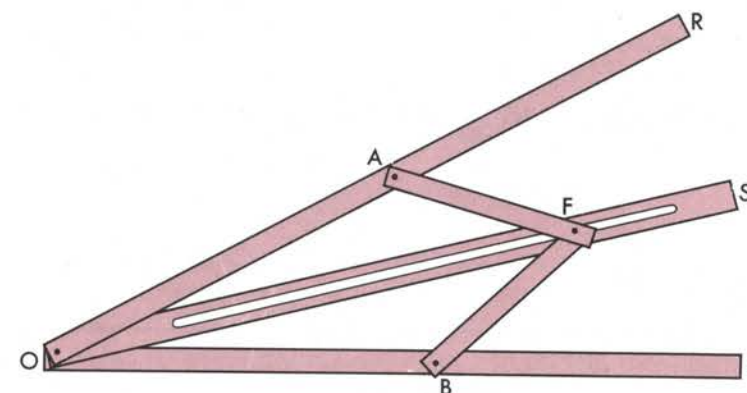

Figure 38

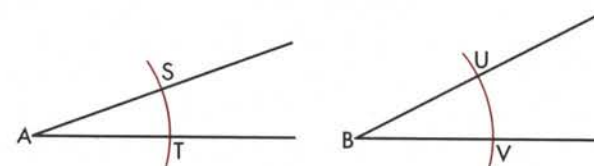**2. Constructing an angle equal to the sum of two given angles**

Figure 39

Draw any working line l and mark any point P. On the vertex A of the given angle A, swing an arc which intersects both sides of the angle at T and S. Using the same compass opening, place the compass point at the vertex B of the given angle B and swing a similar arc locating points V and U. Retaining the same compass opening, place the compass point on point P of the working line l and swing an arc to intersect l at K.
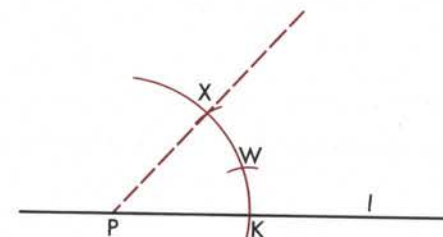

Figure 40

Using a compass opening of length ST (angle A), swing an arc from K on line l, intersecting the previous arc at W.

Using a compass opening of length VU (angle B), swing an arc from W that will intersect the first arc at X. Draw a line through X to P. The angle KPX is the angle $A + B$.

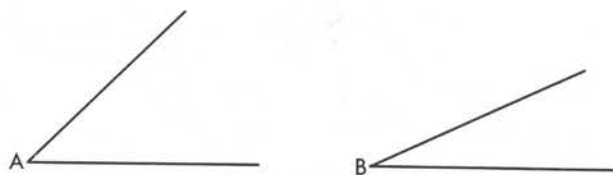### 3. Constructing an angle equal to the difference of two angles

Figure 41

Using a convenient compass opening, place the compass point at the vertex of angle A and swing an arc intersecting both sides of the angle at points M and N. Maintaining the same compass opening, repeat the process for angle B, intersecting the sides of angle B at L and K.

Use the two compass points to determine the size of angle B at points of intersection, L and K. Using this angle size, place the compass point at M of angle A and swing an arc intersecting the
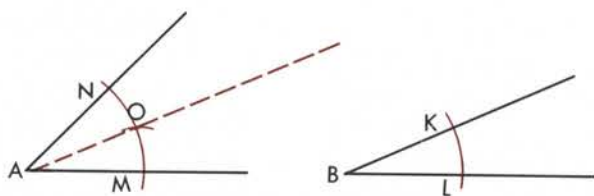

Figure 42

previous arc MN at O. Using point O and the vertex A, draw the straight line AO. The angle OAN is the required angle $A - B$.

4. Constructing the complement and supplement of a given angle

Note: Two angles are complementary if their sum is equal to a right angle. The sum of two supplementary angles is 180°.

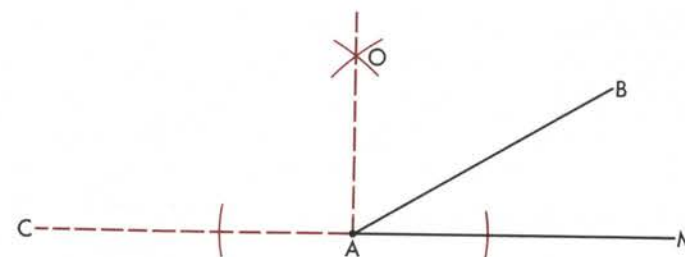To construct the complement of angle MAB, extend MA and

Figure 43

construct a perpendicular OA to point A on line AM. Angle MAO is a right angle, and therefore angle BAO is the required complement if B and O are on the same side of line MA.

Since AC is a constructed extension of MA, the angle MAC is a straight angle (180°), and angle BAC is the required supplement.

Basic Constructions with Triangles

Are you ready to continue your work with basic constructions? You'll find constructions with triangles both interesting and challenging.

Let's begin by considering the construction of a triangle.

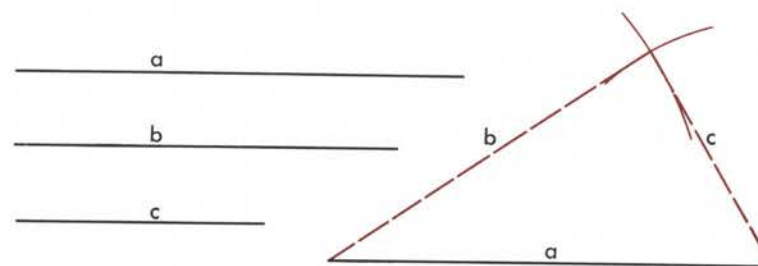### 1. Constructing a triangle with given sides a, b, and c

Figure 44

On given line a, swing an arc with a radius equal to the length of segment b from either end point. From the other end point of line a, swing an arc equal to the length of the segment c. Connect the point of intersection of these arcs with the two end points of line a to obtain the required triangle.

An interesting sidelight of this construction problem is the

restriction put on the lengths of segments *a*, *b*, and *c*. Each line segment must be shorter than the sum of the other two segments if they are to form a triangle.

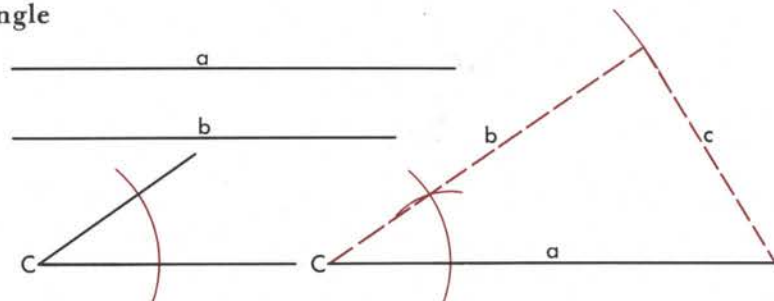**2. Constructing a triangle given two sides and the included angle**

Figure 45

On line *a* and at either end point, construct an angle equal to given angle *C*. From point *C*, swing an arc with a radius equal to segment *b* which intersects the line adjoining side *a*. (Extend this line if necessary.) Connect this point of intersection with the other extremity of line *a* to complete the required triangle.

3. Constructing a triangle given two angles and the included side
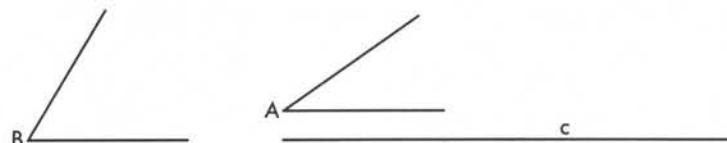
Figure 46

At one end point of line *c* construct the given angle *A*, and at the other end point construct the given angle *B*. Extend the sides
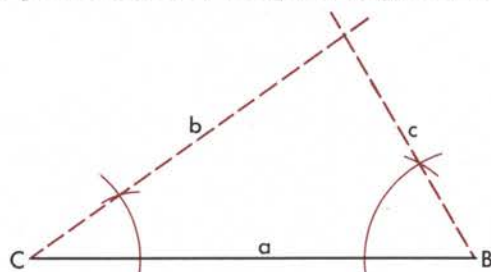

Figure 47

of each angle until they intersect to form the required triangle.

4. Constructing the altitudes and the medians of a given triangle

An *altitude* of a triangle is a line from the vertex of one of the angles and perpendicular to the opposite side. How many altitudes can be constructed in any given triangle? You are correct if your answer is three.

A triangle also has three *medians*. A median of a triangle is a line from a vertex to the midpoint of the opposite side.

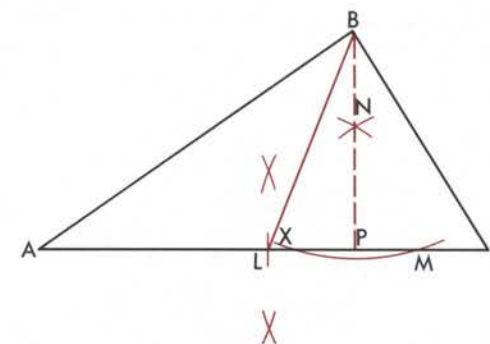Let's construct an altitude from angle *ABC* in Figure 48. Begin

Figure 48

your construction by placing the compass point at point *B* and swinging an arc which intersects line *AC*, or *AC* extended, at *L* and *M*. Using points *L* and *M*, swing arcs which intersect at *N*. Connect points *B* and *N* to complete the required altitude. Our altitude is represented by line *BP*. How many degrees in angles *BPC* and *BPA*?

To construct a median from angle *ABC*, bisect line *AC*. The bisector intersects *AC* at point *X*. The line *BX* is our required median.

Basic Constructions with Circles and Polygons

1. Constructing a circle with a given point *P* as center which passes through another given point *O*

Place one compass point at *P* and open the compass to equal the radius *PO*. Complete the required circle.
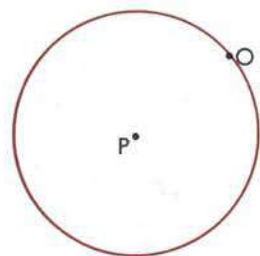
Figure 49

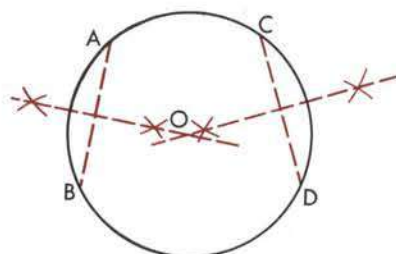## 2. Finding the center of a given circle

Figure 50

Draw any two nonparallel chords, AB and CD, in the given circle. Construct the perpendicular bisectors of chords AB and CD. These bisectors meet at point O, the center of the circle.

3. Bisecting a given arc of a circle

Connect the end points A and B of arc $\overarc{AB}$ with a straight line AB. Construct the perpendicular bisector of AB. The perpendicular bisector also bisects arc $\overarc{AB}$.
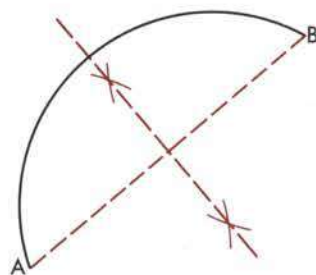

Figure 51

4. Finding the center of a circle given an arc of that circle

The construction is equivalent to that of construction 2. Draw any two nonparallel chords and construct their perpendicular bisectors. The point of intersection is the center of the circle.
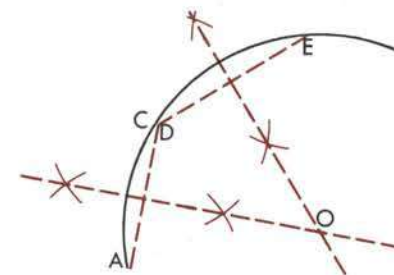

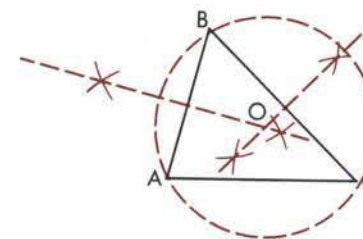Figure 52

5. Circumscribing a circle about a given triangle

Figure 53

Construct the perpendicular bisectors of any two of the sides of the given triangle ABC. They intersect at point O, the circle's center. Using OA, OC, or OB as the radius, complete the required circle.

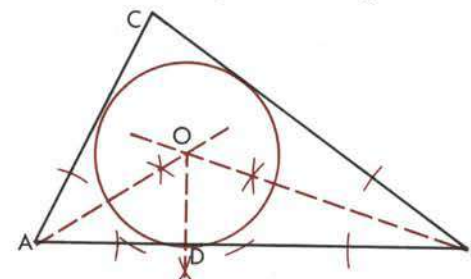## 6. Inscribing a circle in a given triangle

Figure 54

Bisect any two of the angles of the given triangle *ABC*. From point *O*, the intersection of the two bisectors, construct a perpendicular intersecting any side at *D*. Using *O* as center and a radius equal to *OD*, complete the required circle.

7. Constructing tangents to a given circle from a given point outside the circle
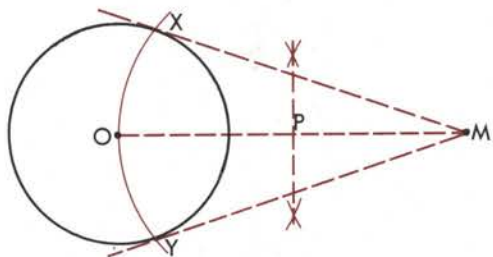

Figure 55

Connect points *M* and *O*, where *O* is the circle's center. Bisect *MO*. Using the midpoint *P* of *OM* as center and *OP* as radius, swing an arc which intersects the circle at *X* and *Y*. Connect points *X* and *M* and points *Y* and *M* to complete the tangents.

> **Note:** The angles *MXO* and *MYO* are right angles as required for tangents and the radii of a circle. In effect, triangles *MOX* and *MOY* are inscribed in semicircles.

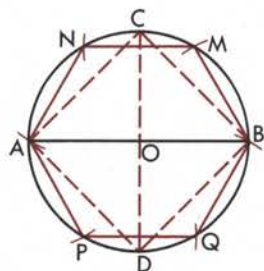**8. Inscribing a square and a regular hexagon in a given circle**

Figure 56

Draw any diameter *AOB* and construct its perpendicular bisector. The perpendicular bisector will intersect the circle at points *C* and *D*. Connect the points of intersection *DB*, *BC*, *CA*, and *AD* to complete the inscribed square.

Using a radius equal to the radius of the circle *OB*, place your compass point at *B* and swing an arc intersecting the circle at *M*. From *M* swing an equivalent arc intersecting the circle at *N*. Continue this process until six points of intersection are evident. Connect these points consecutively to complete the required regular hexagon.

9. Constructing a quadrilateral similar to a given quadrilateral on a given segment as a side
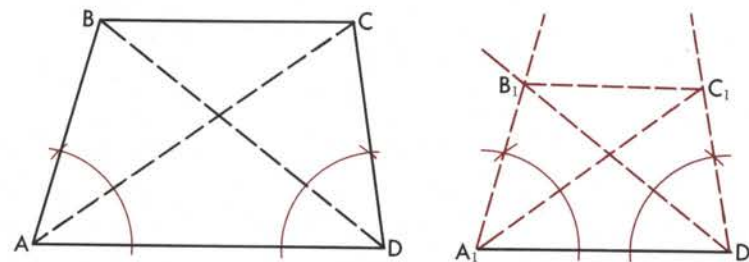

Figure 57

Of the given quadrilateral *ABCD*, draw diagonals *AC* and *BD*. On the given line segment A_1D_1, construct angle *DAB* and then angle *DAC* at point A_1. At point D_1, construct angle *ADC*, then angle *ADB*. A_1C_1 intersects D_1C_1, and D_1B_1 intersects A_1B_1 in the required proportions. Connect points B_1 and C_1 to complete the required construction.

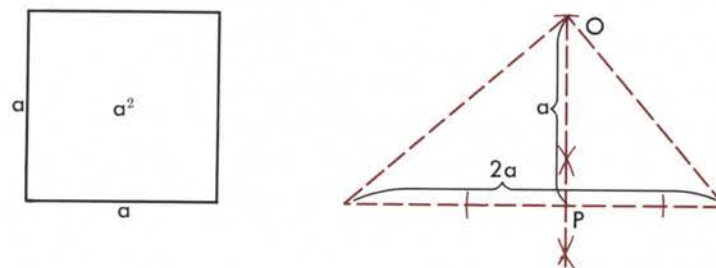**10. Constructing a triangle equal to the area of a given square**

Figure 58

Construct the base of the triangle equal to twice side *a* of the given square. At some point *P* on the base, construct a perpendicular *PO* of length *a*. Connect each extremity of the base line with point *O* of this perpendicular.

The area of a triangle equals $\frac{1}{2}bh$. In our completed construction, the area equals

$$\frac{1}{2}(2a)(a) = \frac{1}{2}(2a^2) = a^2$$

11. Constructing a circle, given any three noncollinear points, which passes through each given point
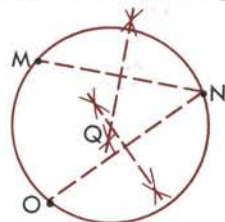

Figure 59

Begin the construction by connecting points M and N and points N and O. Construct perpendicular bisectors of each line MN and NO. Using the point of intersection Q of these perpendiculars as center and QN as radius, complete the required circle.

EXERCISE SET 3

Basic Constructions Revisited

Complete the following constructions.

1. Construct a line through a given point P which forms equal angles with the sides of a given angle.

Note: Bisect the given angle. Next, construct a perpendicular from point P to the angle bisector. Continue the perpendicular so that it intersects the sides of the given angle at B and C. Are angles C and B equal?

2. Construct the quantity $a(b + c)$.

Note: You should recall our earlier work in constructing the quantities $a + b$ and $a \times b$. Use this knowledge to complete the required construction.
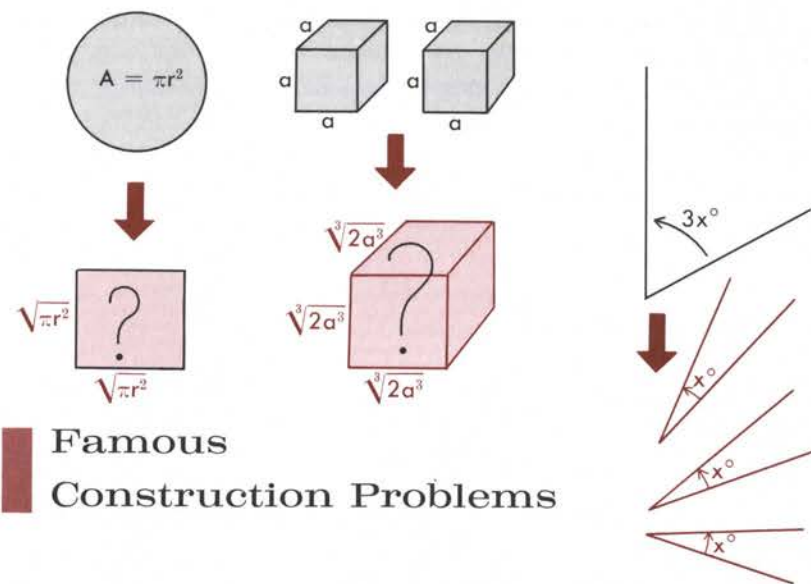
3. Construct a triangle with:

a. All sides equal   b. Two sides equal   c. No sides equal

4. Given two angles of a triangle, construct the third angle of the triangle. (What is the sum of the angles of a triangle?)

5. Circumscribe a hexagon about a given circle.

6. Given circle O and a point P outside the circle, construct two tangents to the circle which pass through point P.

Famous Construction Problems

Duplication of the Cube

You explored earlier the problem of constructing a cube that is twice the volume of a given cube. You found that an attempt to double the volume by doubling the sides of the cube resulted in a volume eight times that of the given cube. Why can't we solve this problem by construction? Let's examine this idea in more detail.
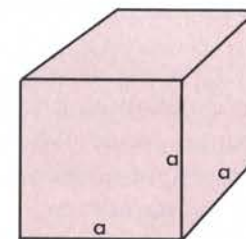
Suppose we are given a cube with sides equal to segment a.

Figure 60

You find the volume by the formula $V = a^3$. If you are to double the volume of a^3, you must construct a segment b of the required cube in such a way that $b^3 = 2a^3$. If you assign the value of 1 to a of the given cube, then $b^3 = 2$ and $b = \sqrt[3]{2}$. At this point you

must recall the important key to constructibility explored on pages 5 to 7. Construction by compass and straightedge alone is possible only when our construction involves arithmetic operations or the extraction of real square roots. Since the expression $b = \sqrt[3]{2}$ cannot be expressed according to our criteria, the duplication of the cube is impossible to construct by straightedge and compass.

Can a Circle Be Squared?

The most famous construction problem is that of squaring the circle. Do you think it can be done? Suppose you are given a circle with radius r. What is the area of the circle? Since the area of a circle is found by multiplying π (pi) by the radius squared, the area of our circle is πr^2. Suppose you assign the value of 1 to radius r. What is the area of the circle? Since $1^2 = 1$ and $1 \times \pi = \pi$, our circle has an area equal to π, and $A = \pi$.

Our problem is to construct a square equal in area to π. If we let b represent the side of the square, than $b^2 = \pi$, and $b = \sqrt{\pi}$. You'll find this impossible to construct. Since π cannot be expressed according to our criteria for constructibility, we cannot complete the construction by compass and straightedge. It is not possible to express $\sqrt{\pi}$ in a finite number of rational operations and real square roots as required by our rules of constructibility.

Trisecting an Angle — Another Challenge

You'll recall that the bisecting of any given angle was quite easy. It would seem that dividing an angle into three equal parts by straightedge and compass would not be much more difficult. However, not only is trisecting a given angle by straightedge and compass more difficult, it is impossible.

Of course you could trisect certain specific angles such as 135° or 90°. But given any angle A, it is impossible to trisect it without violating the rules. Why? It can be shown that the algebraic equation, sometimes called the *trisection equation*, is a cubic equation in the form $A^3 - 3A - 2B = 0$. It can be shown again that the equation has no roots which can be expressed according to our criteria for constructibility.

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Making the Impossible Possible — Two Points to Remember

You will violate the rules of the game of construction if you use devices other than those allowed. Even one point marked on your straightedge is not permissible.

You'll see how two points marked on the straightedge can make the impossible trisection problem possible. In Figure 61, straightedge S is shown with two points T and U marked on it. The distance between points T and U is represented by $2a$.
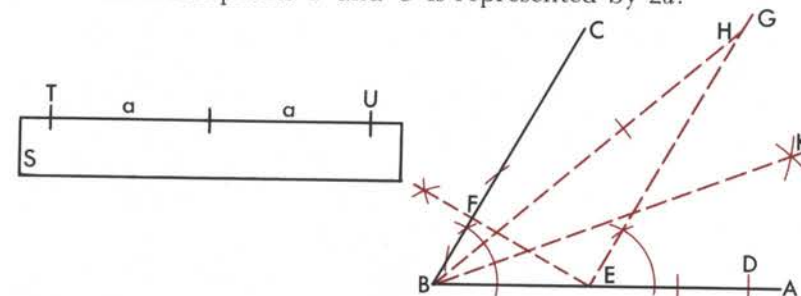

Figure 61

Given any acute angle ABC, on line segment AB lay off segment BD equal to the length $2a$. From the midpoint E of BD, construct a perpendicular to BC, intersecting BC at F. Next, construct a line segment EG parallel to BC through point E. Now lay your marked straightedge so that it passes through the vertex B of angle ABC, such that the points T and U lie on lines EF and EG respectively. Call the point where U lies on line EG point H. Draw BH. Now bisect angle HBA with line BK. Since it can be proven that angles CBH, HBK, and KBA are equal, BH and BK trisect angle ABC.

> **Note:** If the given angle happens to be an *obtuse* angle (an angle greater than a right angle and less than a straight angle), this method will trisect the *acute* angle formed by the supplement of the given angle. With the supplement trisected, it is possible to derive the trisection of the given angle.
>
> Let x be the measure of an angle greater than 90°; $180° - x$ is the measure of its supplement. If $\frac{180° - x}{3} = k$ (one-third of the supplement of x), then $x = 180° - 3k$. We wanted to find one-third of x, so $\frac{x}{3} = 60° - k$. It is easy to construct a 60° angle by constructing an equilateral triangle (each angle of an equilateral triangle is a 60° angle). Thus by subtracting angle k from a 60° angle we will have trisected the obtuse angle.

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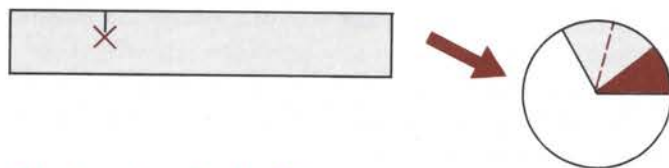
X Marks the Spot — Angle Trisection

Let's consider another method of trisecting an angle. This time our straightedge will have only one mark, and again we violate the rules to complete the construction. However, the solution is interesting, so let's proceed.
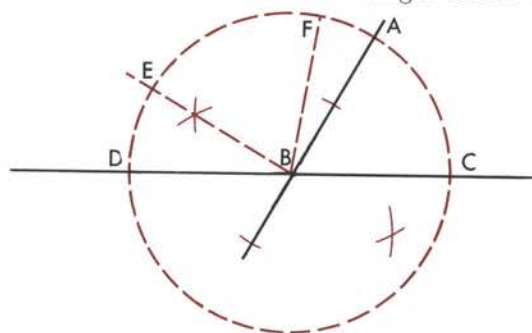
Our problem is to trisect the acute angle ABC. Using BC as

Figure 62

radius and point B as center, construct a circle. Extend radius BC to the point D. Line DBC is a diameter of the circle. Construct a $90°$ angle with line BA as one side and line BE as the other. Mark a point equal to the circle's radius from the end of your straightedge. With the end of the straightedge on the circumference, move it so that the mark lies on line BE and point D lies along the straightedge also. Connect point B and the point F on the circumference coinciding with the end of the straightedge. Now, $\angle ABF = \frac{1}{3}\angle CBA$. Our proof of the construction is based upon a series of steps showing that arc $\overset{\frown}{FA}$ of the circle equals one-third of the arc intersected by $\angle CBA$. We can conclude that $\angle ABF = \frac{1}{3}\angle CBA$.

Although this method will only work for acute angles, it is possible to use this method to trisect any obtuse angle. If an angle is obtuse, its supplement is acute. Thus it is only necessary to trisect the supplement of an obtuse angle. After trisecting the acute angle, it is simple to construct the trisection of the obtuse angle by means of the method illustrated in the preceding section.

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Some Can, Some Can't — Constructing Regular Polygons

You have already constructed an equilateral triangle and a regular hexagon. To construct a six-sided figure with equal sides, you use a compass opening equal to the radius of a given circle and mark six consecutive points on the circle's circumference as in Figure 63.
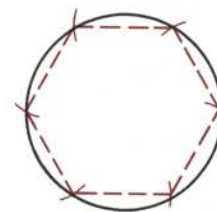

Figure 63

Can regular polygons of 5, 6, 7 or 8 sides be constructed? Let's consider the construction of a regular pentagon, a five-sided figure whose sides are equal. One common method used to complete the construction is as follows.

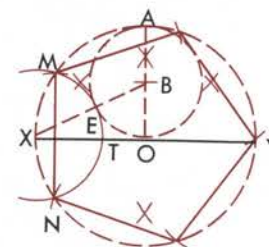Construct a circle for any diameter XY as in Figure 64. Construct

Figure 64

the perpendicular bisector of XY which will intersect the circle at point A and line XY at point O. Bisect AO and using its midpoint B as a center, construct a circle of radius BO. Connect point X and point B. Line XB intersects circle BO at point E. Using point X as center and length XE as radius, swing an arc which intersects circle O at points M and N and line XY at point T. Use MN as the compass opening to mark five consecutive points on circle O. Connect each of the five points with straight line segments to complete the required pentagon.
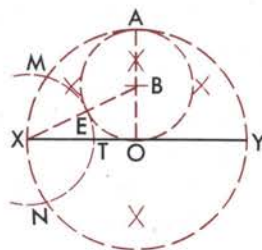
43

Figure 65

A regular decagon is a 10-sided figure. By bisecting each side of the pentagon in Figure 65, the construction of a regular decagon can be completed. Five points are determined where each bisector intersects the circle's circumference. Use these five points and the five points of the pentagon to complete your decagon.
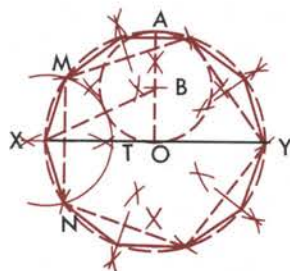

Figure 66

Refer to Figure 65 once again. Bisect one side of the pentagon. Use your compass to measure one-half the bisected segment. Compare line XT in Figure 65 with the prior measurement. What can you learn from it? You can see that line XT could serve to divide the circle into 10 equal parts. Do you see that line XT also bisects line MN? Remember that points M, T, and N are all of equal distance from point X. $XN = XM = XT$. And points M, X, and N are three established points for vertices of our regular decagon.

In 1796, when he was only nineteen, Karl Frederick Gauss surprised the mathematical world by proving that a 17-sided regular polygon could be constructed. Gauss stated that if S is a prime number in the form $S = 2^{2^n} + 1$, when n is an integer, then a regular polygon of S sides could be constructed.

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Let's explore this idea in more detail. You have already constructed a regular polygon of five sides. Insert the value of 1 for n into the equation,

$$S = 2^{2^n} + 1.$$
$$S = 2^{2^1} + 1 = 2^2 + 1 = 5$$
$$S = 5$$

We see that $S = 5$. Since 5 is a prime number, a regular pentagon can be constructed.

Let's try another value for n, say 2. The equation becomes

$$S = 2^{2^2} + 1 = 2^4 + 1 = 17.$$

Therefore we know that a regular pentagon of 17 sides can be constructed.

If we let $n = 3$, then the prime number 257 results. We can conclude that a regular polygon with 257 sides is possible.

A regular polygon construction with 7 or 11 sides is impossible. We are not able to find a number n such that $7 = 2^{2^n} + 1$.

EXERCISE SET 4

Exploring Famous Constructions

1. Using the equation $S = 2^{2^n} + 1$, show that S is prime when $n = 3$.

2. Using $n = 0$, 1, 2, 3, and 4, find S for the equation $S = 2^{2^n} + 1$. Which of the values for S are prime numbers?

3. Can the area of a square be doubled? Examine Figure 67 below.
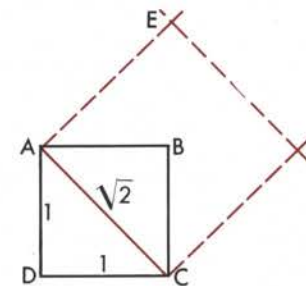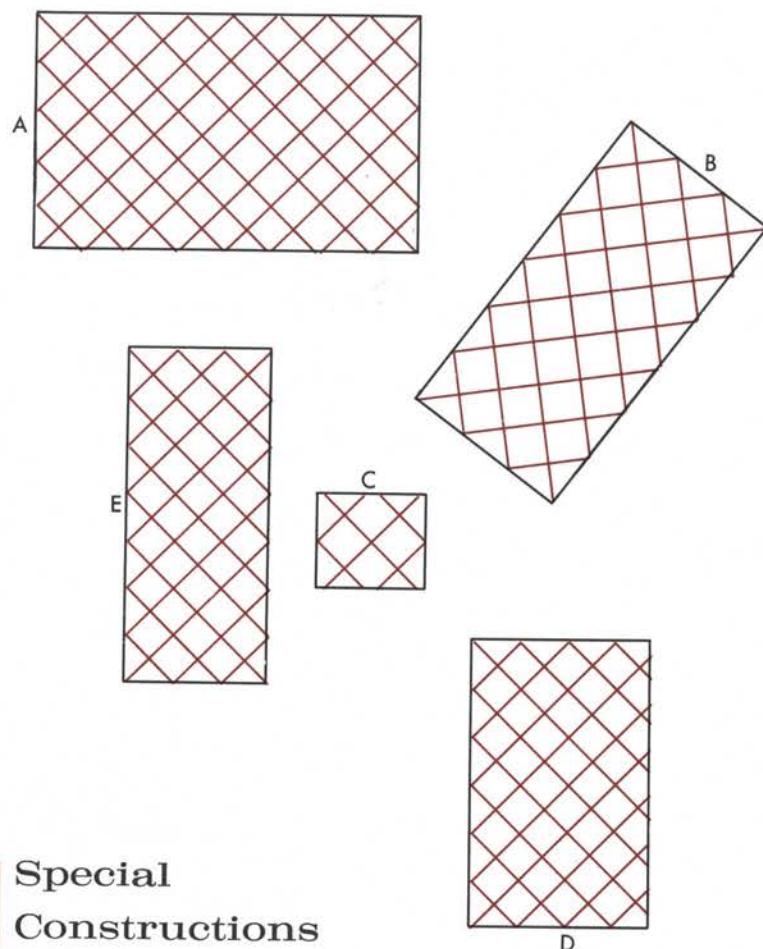

Figure 67

What is the value of line AC? What is the area of square $AEFC$? Of $ABCD$? Is the area of square $AEFC$ double that of square $ABCD$?

4. A given cube has sides of 2 inches. By doubling each side of the cube, what happens to the volume as compared with the original cube?

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Special Constructions

The Golden Section — The Divine Proportion

You've heard the expression, "easy on the eye." Take a look at the five rectangles above. Which rectangle is most pleasing to your eye?

Did you choose rectangle A or rectangle D? Both rectangles A and D have the proportions which have been found to be the most pleasing to the human eye.

The early Greeks discovered what they called the *divine proportion*, or *golden section*. They found that the line most pleasing

to the eye was one divided in an approximate ratio of 5 to 8. The line in Figure 68 uses this ratio.

Figure 68

According to the divine proportion,

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

We say that the line is divided into mean and extreme ratio. You can divide any given line into mean and extreme ratio by using the following construction. We used this idea earlier in our construction of a regular pentagon.
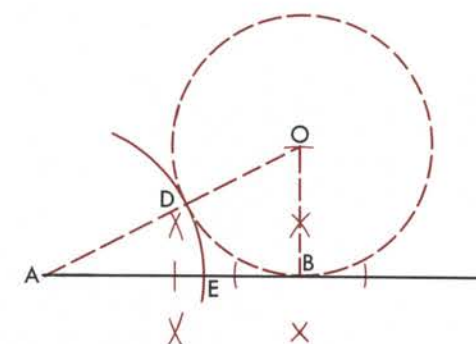
Figure 69

Given a line AB, first construct its bisector. At end point B of line AB, construct BO perpendicular to AB and equal to one-half of AB. Construct a circle with O as the center and one-half of AB as a radius. Draw a line between points A and O that intersects the circle at D. Swing an arc from A equal to AD to intersect AB at E. Now point E divides line AB into mean and extreme ratio.

Let's explore some interesting ideas about the golden section. The numerical value of the ratio is found to be close to .618.

Suppose you consider AB to be the length of one side of a rectangle and AE to be the width of the rectangle. According to the ancient Greeks, the golden rectangle is one whose width is to its length as its length is to one-half the perimeter. Thus, in the golden rectangle in Figure 70, $\dfrac{BC}{AB} = \dfrac{AB}{AB + BC}.$
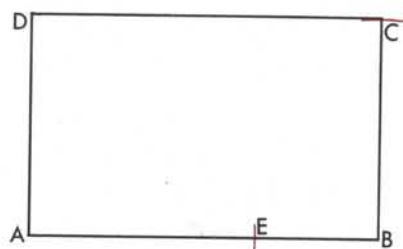
Figure 70

Let's assign the value of 1 to length AB. Our proportion then becomes, $\dfrac{BC}{1} = \dfrac{1}{1 + BC}$.

By algebra:

$$\frac{BC}{1} = \frac{1}{1 + BC}$$
$$BC + (BC)^2 = 1$$
$$(BC)^2 + BC + \frac{1}{4} = \frac{5}{4}$$
$$(BC + \frac{1}{2})^2 = \frac{5}{4}$$
$$BC + \frac{1}{2} = \frac{1}{2}\sqrt{5}$$
$$\therefore BC = \frac{1}{2}\sqrt{5} - \frac{1}{2} = .618$$

You can see, then, that in order to determine the width of a golden rectangle if the length is known, you multiply the length by .618. If you are given the width you can find the length by dividing the width by .618. Try it! What is the length of a golden rectangle whose width is 5 inches?

It is interesting that the golden section is found in both man-made objects and in nature. The divine proportion has been widely used in art, architecture, and design. Nature reveals the golden ratio in many instances. You can find the principle of the golden section in living things such as starfish, snails, and other animals. The golden section is also found in the arrangement of seeds in flowers, in the arrangement of leaves on the stems of plants, and in certain geological formations.

Many great artists, such as Michelangelo, Botticelli, and Da Vinci, used the golden section in their masterpieces. You have no doubt read about the great Parthenon in Athens, Greece. Many authorities say this structure is perfectly designed. The principle of the golden section is evident throughout.

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The golden section has influenced design throughout the world. In fact, you'll find that the size of this booklet approximates the golden rectangle.
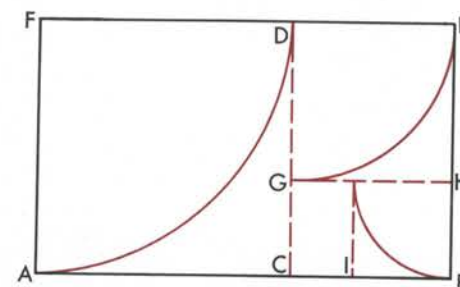

Figure 71

Suppose you are given the golden rectangle as in Figure 71. By swinging side FA as shown and dropping a line from D to C, a new golden rectangle $DEBC$ is formed. Now let's swing side DE in a similar manner. A third golden rectangle — $BHGC$ — is formed. We could continue this process indefinitely. Each time we form a new rectangle it has the same golden ratio as the previous one.

Draw any line m and a second line n. Draw a third line o, equal to the length $m + n$. Finally, draw a fourth line p equal to $n + o$.
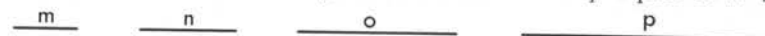

Figure 72

Now, on any horizontal working line a, swing an arc from an end point A equal to length p as shown in Figure 73. This locates point B. From B, swing a second arc of radius o to intersect the first arc at C. Swing the same arc from points A and B to intersect
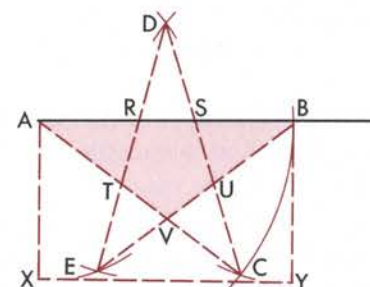

Figure 73

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at D. Connect points D and C and A and C. From point A swing an arc equal to length o below point A and from point D an intersecting arc of length p. These arcs meet at E. Connect points D and E and B and E. You have completed a magnificent figure! Note the length of line TR. Compare it with the original line m. Now check the length of lines TD and VC. How do their lengths compare with lines o and n in Figure 72?

Triangle ABV in Figure 73 is called the golden triangle. From points A and B in Figure 73, complete the rectangle $ABXY$. You might be surprised to learn that this rectangle is a golden rectangle.

The Snowflake Construction

You probably have seen many interesting designs that use geometric figures. Perhaps you have made such a design. Many times we don't really stop to think about the mathematics involved in a design, in architecture, or in many aspects of nature.

Next we'll consider a construction commonly called the *snowflake construction*. You'll see how it received its name later on.

Let's begin the construction by completing a triangle with three equal sides. Next, each side of the triangle must be divided
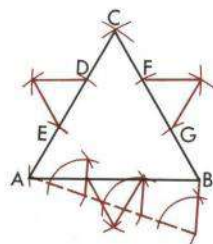
Figure 74

into three equal parts. This can be done by using the basic construction for dividing a line into any number of equal parts. You explored this earlier. Using the two points of trisection on each side, construct an equilateral triangle. If you erased all lines except those on the perimeter of the figure, what figure would result? You can see in Figure 75 that you would have a six-pointed star. Sometimes this star is referred to as the Star of David.
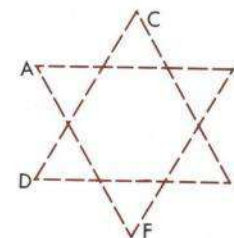
Figure 75

You might have discovered another convenient method for constructing this star. By constructing a circle and inscribing a hexagon, the star is obtained by connecting every other point. Thus, in Figure 75, we connect points CD, DE, and EC. Then, connect AB, BF, and FA to form the star.

Repeat the construction of an equilateral triangle on each of the 12 sides of the Star of David. After you have done this, your figure looks like the one in Figure 76.
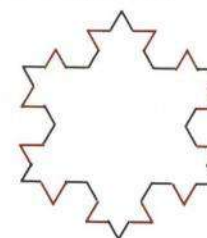
Figure 76

In carrying out the snowflake construction, you must use a good compass, straightedge, and sharpened pencil. If you continue the construction process of completing an equilateral triangle on each of the 48 line segments of the construction in Figure 76, your construction will look more and more like a snowflake. After each construction, you have four times the number of line segments you had before. You began the snowflake construction with a three-sided figure. After completing the six-pointed star, you had a figure of 12 sides, or 3×4 sides. From the star of 12 sides, you constructed Figure 76 of 48 sides, or 12×4 sides. What would be the number of sides if the construction were repeated on the 48-sided figure? You are correct if your answer is 192. You might want to complete a snowflake construction for a class project.

The Nine-point Circle

You have explored many circles in the work thus far. But let's consider one of the best-known circles in mathematics, the *nine-point circle*. Can you visualize a circle which passes through nine specific points?

Let's begin our construction of the nine-point circle by constructing the midpoints of the three sides of any given triangle (Figure 77). These are points D, E, and F. Next, construct the three altitudes of the triangle by constructing perpendiculars from
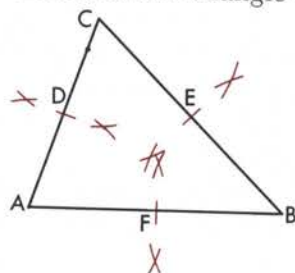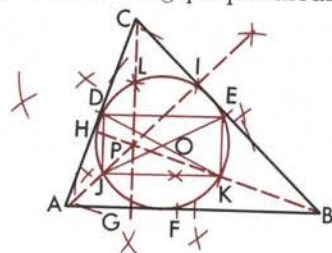

Figure 77

Figure 78

each of the vertices A, B, and C to the opposite sides. These altitudes are perpendicular at points G, H, and I on the triangle. These three altitudes meet at point P.

You now have six points on the triangle, D, H, E, I, G, and F. To locate the final three points, bisect the three lines AP, CP, and BP. Mark their midpoints J, K, and L.

Now that the nine points are located, how can the circle be determined? You could find the circle's center by the three-point method explored earlier. A quick method of finding the lost center is to complete a rectangle using points D, E, K, and J. Construct the diagonals of the rectangle, and their intersection determines the circle's center, point O.

Use point O as center and any one of the other nine points as a radius, and complete a circle. Each one of the nine points lies on the circumference of the circle.

There are many mathematical ideas connected with the nine-point circle. We cannot explore all the interesting relationships; many of these are too advanced. You should, however, note a few of these relationships. For example, the center of the nine-point circle could have been located by finding the midpoint of the line joining the intersection of the altitudes and the intersection of the

bisectors of the sides of the triangle. You could have found the center by constructing perpendicular bisectors of any two non-parallel chords of the circle.

Note that line FL is a diameter of the nine-point circle. Also, angle DHB is a right angle. These are just a few of the relationships of the nine-point circle. Can you discover others?

Hexagons and Honeycombs

Many of the works of Mother Nature reflect the works of mathematics. Did you ever examine the bee's honeycomb and notice that each individual section is a regular hexagon? The construction of a honeycomb is extremely interesting. You have already constructed a hexagon. Let's begin by repeating this construction (Figure 79). The hexagon represents only one section
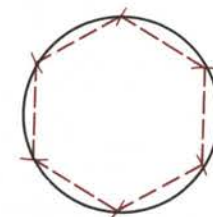

Figure 79

of our honeycomb. Let's proceed to add additional sections. On any single side of the original hexagon, construct an equilateral triangle with vertex O, as in Figure 80. Using point O as center and
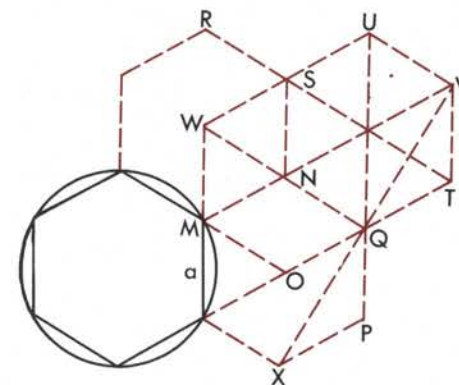

Figure 80

a compass opening that is equal to a side of the triangle, complete a circle. In circle O, inscribe a second hexagon. Repeat the process of constructing an equilateral triangle on side MN of hexagon O and proceed to complete another circle and hexagon. The honeycomb is now beginning to take shape.

You can see that the construction processes could be repeated to form as large a honeycomb as desired. Actually, your work could be reduced at this point in the construction by taking advantage of certain relationships which exist. For example, to complete the fourth section of the honeycomb in Figure 80, you can extend lines MN, PQ, and RS. Note that these three lines meet in the center of the fourth hexagon. OQ extended intersects RS extended at T, and WS intersects the extension of PQ at U. Extension MN and XQ intersect at the last point V of the hexagon. If you continue this process, your honeycomb can have as many hexagonal sections as desired.

Just for Fun

In Figure 81, you are given angle ABC with line OL through the vertex B. Construct a triangle EFG with its vertex F on line OL and vertices E and G on the sides of the angle.
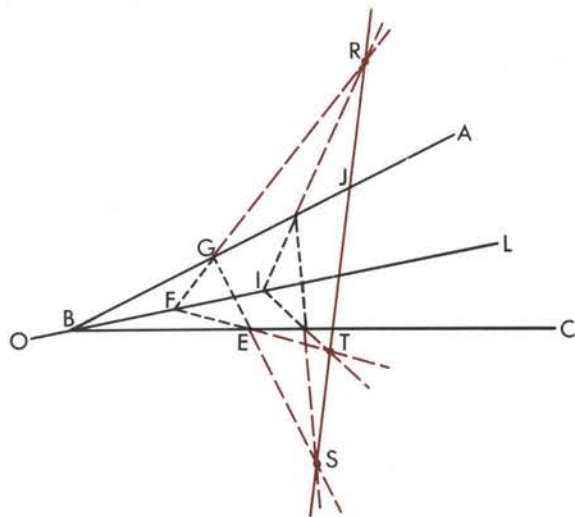
Figure 81

Construct a second triangle HIJ with vertices placed similarly. Extend line FG to meet extended line IJ at R; extend line JH to meet extended line GE at S; and extend FE to meet extended line IH at T. What do you notice about points R, S, and T? Connect these points. Note that R, S, and T form a straight line.

Construct a third triangle and extend its sides as you did before. What do you note about the extended lines? You have demonstrated the well-known *Desargue's theorem*. We say that the two triangles are in *perspective*. You found that corresponding sides of triangles in perspective meet in three points which lie on one straight line. We say the points are *collinear*.

EXERCISE SET 5

More Construction Fun

1. Construct a square equal to the sum of two given squares. (**Hint:** If a and b are the sides of the two known squares and c the side of the required square, then $a^2 + b^2 = c^2$. Side c will be the hypotenuse of a right triangle whose legs are equal to a and b respectively.)

2. Construct a square whose area is equal to the given parallelogram $MNPQ$. (**Hint:** Using your knowledge of the construction of a mean proportional, construct a mean proportional between lines MN and PQ. On working line l, construct a line AF such that $AD =$
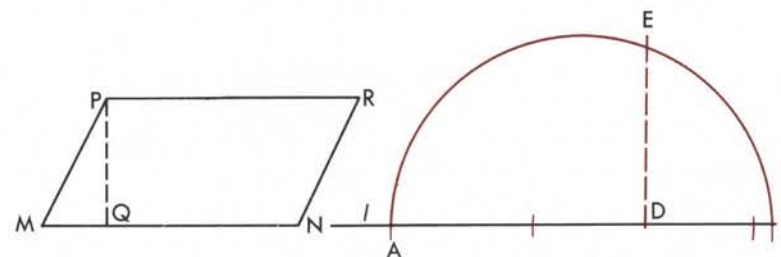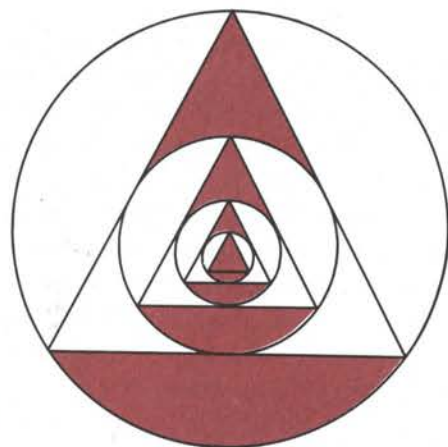

Figure 82

MN and $DF = PQ$. In Figure 82, the area is equal to $MN \times PQ$, $\dfrac{AD}{ED} = \dfrac{ED}{DF}$, and $(ED)^2 = AD \times DF$ or $MN \times PQ$. Complete the required square.)

Construction and Designs

Beauty in Lines and Circles

You don't have to be an artist to make artistic designs. Your knowledge of basic constructions can be used for bringing beauty to lines and circles.

Examine Figures 83, 84, and 85. How many different geometric figures can you recognize in these designs?
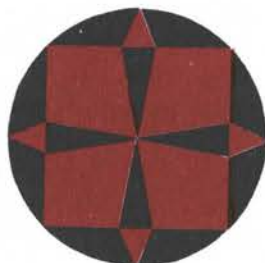

Figure 83   **Figure 84**   **Figure 85**

Let's consider another design. This design is made up of curved lines only. Begin the design by completing a circle. Using the radius of the circle as the compass opening, mark six points on the circumference of the circle. Place your compass point at each of the six points, and swing arcs to form six symmetrical "leaves" of the design. By locating a point midway between any two of the known points, you can use it to locate five other "midpoints" on the circumference. Use these points to swing arcs which form six more "leaves" of the design. From each of the six original points, swing arcs of a length slightly less than the radius of the circle.

Study the design in Figure 88. Can you describe the construction needed to complete the design? Note the similarities between this design and Figures 86 and 87.

Figure 86   **Figure 87**   **Figure 88**

An interesting straight-line design can be completed by using parallel lines. You begin by constructing a *rhombus*. A rhombus is a geometric figure with adjacent sides equal. By placing a line at point A making any convenient angle with line AB, we can construct a rhombus quickly. Construct a line equal to line AB on the line forming an angle with AB. From point B construct a line parallel to the first line AC. Construct BD equal to AB. Connect points C and D.

Consider the rhombus in Figure 89. You can observe that line EF has been constructed parallel to AB and CD. Note that points E and F bisect lines AC and BD.
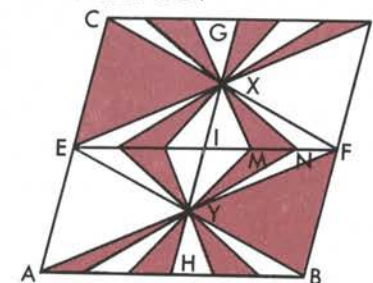

Figure 89

Also, line GH bisects lines CD, EF, and AB. Using your knowledge of dividing a line into equal parts, trisect line IF. Use the distance IM to trisect each line CG, GD, EI, AH, and HB. Next, bisect lines GI and IH at X and Y. Connect N and X, M and Y, and N and Y. Repeat this process, using the other points of trisection and the vertices of the rhombus. By shading or coloring in various ways, the construction becomes a fascinating design.

Nineteen Circles in a Circle

On any circle, construct a perpendicular bisector of any diameter of the circle. Extend the perpendicular so that it intersects the circle in two points. Bisect two of the adjacent right

angles and extend the bisectors so that four other points are determined on the circumference. Connect consecutive points.

Now bisect one side of the octagon. Use a compass opening equal to one-half the side and the eight vertices as centers to complete eight equal circles. By using the center of the original circle, do you see how two circles can be constructed which touch
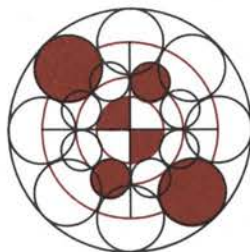

Figure 90

each of the eight equal circles? By repeating the construction process, eight additional equal circles can be formed. With the use of shading, our construction becomes a fascinating design.

Turning Squares into Designs

The design in Figure 91 is based on a construction of "turned squares." You simply construct a first square and then a second square so that the vertices of the first are midpoints of the sides of the second square. Continue the turning of squares until your figure looks like that in Figure 91. By using different shadings, your squares can be turned into many different designs.
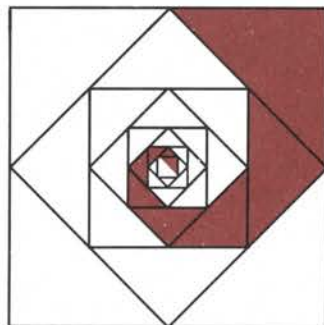
Figure 91
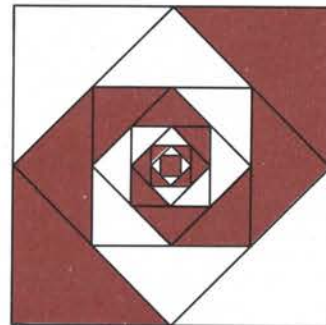
Figure 92

A Backward Glance and a Look to the Future

In exploring the topic of geometric construction on your own, you have found that construction is fun as well as practical.

Construction with straightedge and compass has challenged the minds of men since ancient times. Such famous construction problems as duplicating the cube, trisecting an angle, and squaring a circle now are known to be impossible. Only by violating the rules of the game can some constructions be completed.

You have learned the importance of having a good knowledge of basic constructions. Such constructions as bisecting an angle, constructing perpendiculars, dividing lines into equal parts, and dividing lines proportionally are used again and again in complex constructions and in making designs.

You'll find your work in this booklet of value to you as you progress in the study of mathematics. Your knowledge will help you to see relationships in geometric figures. This work should also give you a new appreciation and awareness of geometric forms in nature, art, architecture, and design.

Upon completing the study of this booklet, you are ready to explore other ideas in geometric construction. The references which follow will provide you with other resource material.
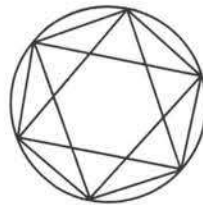
For Further Reading

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YATES, ROBERT C., *Geometrical Tools*. Educational Publishers, Incorporated, 1949.

YOUNG, J. W. A., Editor, "Constructions With Ruler and Compasses: Regular Polygons," Chapter VIII, L. E. Dickson in *Monographs on Topics of Modern Mathematics*. Longmans, Green and Company, 1932.
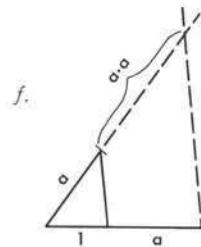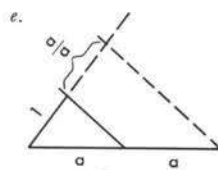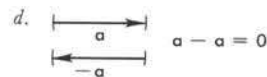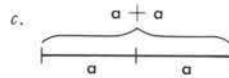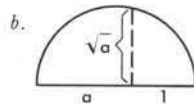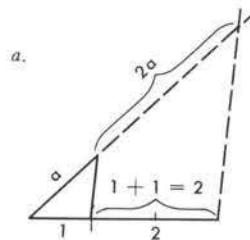
Solutions to the Exercises

EXERCISE SET 1

1. Construction by straightedge and compass is possible only when the construction involves a finite number of the arithmetic operations of addition, subtraction, multiplication, division, and the extraction of real square roots. This criterion serves as a mathematical proof as to the possibility or impossibility of a particular construction.

2. A regular hexagon is formed.
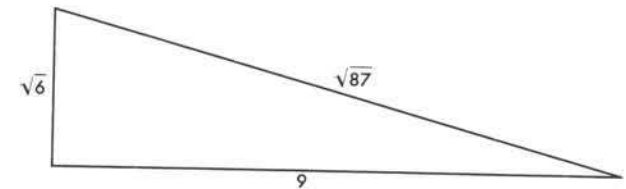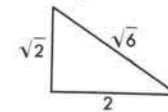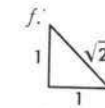
3. A six-pointed star is formed. Thirty-two triangles can be counted.

EXERCISE SET 2

1. Given: $\underline{\quad 1 \quad}$ $\underline{\quad a \quad}$

a. $2a$, a, $1 + 1 = 2$, 1, 2

b. $\sqrt{a}$, a, 1

c. $a + a$, a, a

d. a, $-a$, $a - a = 0$

e. a/a, 1, a, a

f. $a \cdot a$, a, 1, a

2.

a. 1, $\sqrt{2}$, 1; 1, $\sqrt{3}$, $\sqrt{2}$; $\sqrt{3}$, $\sqrt{19}$, 4

b. 1, $\sqrt{2}$, 1; $\sqrt{2}$, $\sqrt{6}$, 2

c. 1, $\sqrt{2}$, 1; $\sqrt{2}$, $\sqrt{18}$, 4

d. 1, $\sqrt{26}$, 5

e. 1, $\sqrt{2}$, 1; $\sqrt{2}$, $\sqrt{11}$, 3

f. 1, $\sqrt{2}$, 1; $\sqrt{2}$, $\sqrt{6}$, 2; $\sqrt{6}$, $\sqrt{87}$, 9

3. Given: $\underline{\quad 1 \quad}$ $\underline{\quad a \quad}$ $\underline{\quad b \quad}$ $\underline{\quad c \quad}$

a. $a + b$; a, b; $b + a$; b, a

b. $b + c$; b, c; $c + b$; c, b

c. $a + (b + c)$; a, $(b + c)$; $(a + b) + c$; $(a + b)$, c

d. $a \cdot c$; c, 1, a; $c \cdot a$; a, 1, c

e. $a(b + c)$; $a \cdot b + a \cdot c$; $a \cdot b$, $a \cdot c$; $a \cdot c$; $a(b + c)$; $(b + c)$; c, 1, a; $a \cdot b$; b, 1, a
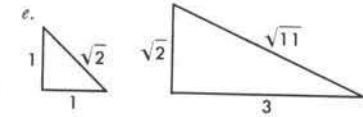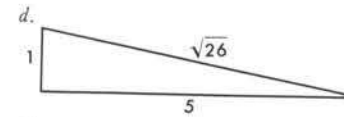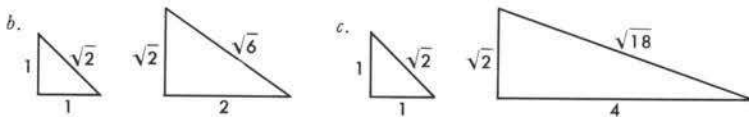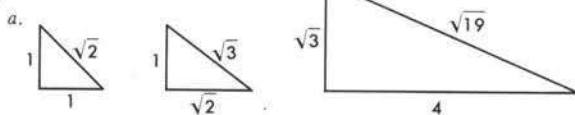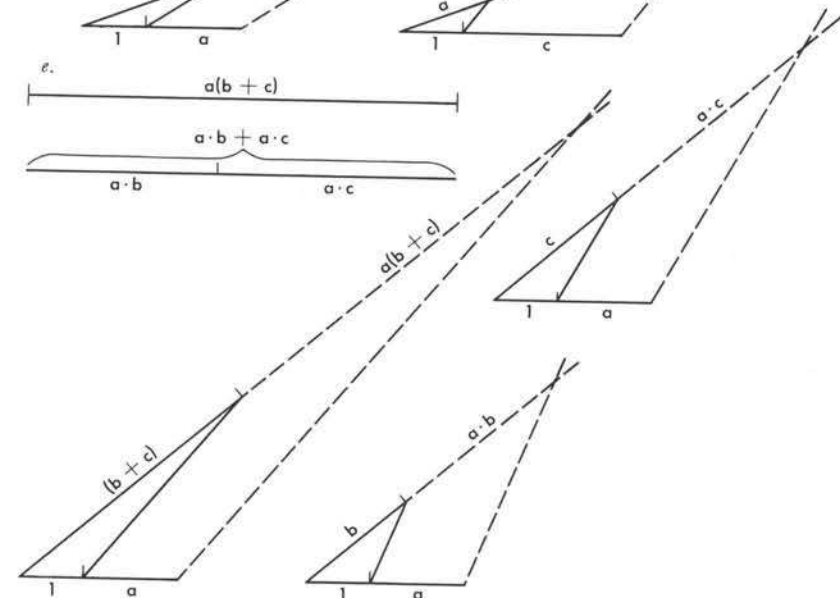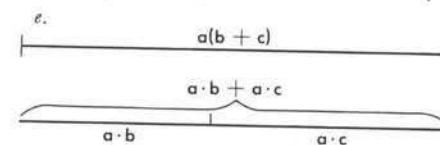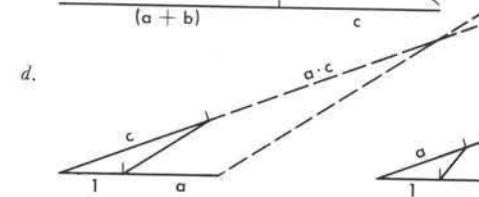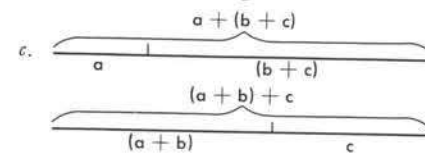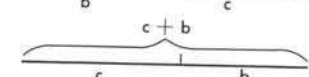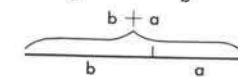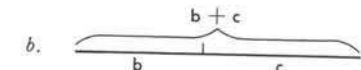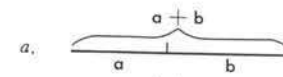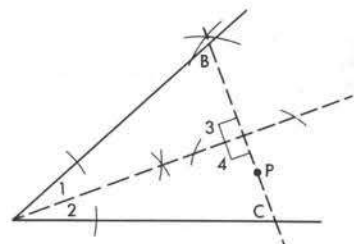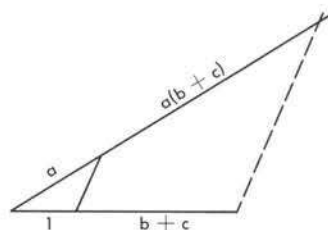
4. *a.* commutative principle of addition
   b. commutative principle of addition
   c. associative principle of addition
   d. commutative principle of multiplication
   e. distributive principle for multiplication with respect to addition

EXERCISE SET 3

1.   2.

Since $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ by construction, $\angle B$ and $\angle C$ must be equal. $\angle 1 + \angle 3 + \angle B = 180°$ and $\angle 2 + \angle 4 + \angle C = 180°$, $\therefore \angle B = \angle C$

3. *a.* Draw any base line *a*. From each of its endpoints, swing arcs of radius *a* which intersect. Use the point of intersection and the two endpoints of line *a* to complete the equilateral triangle.

   b. Draw any base line *a*. From each of its end points, swing arcs of a radius greater than $\frac{1}{2}a$. At the point of intersection of the arcs, connect the point with each end point of line *a* to complete the isosceles triangle.

   c. Draw any base line *a*. Using any radius not equal to *a*, swing an arc from one end point of line *a*. Using a different radius and one not equal to *a*, swing an arc which intersects the previous arc. Use the point of intersection and the two endpoints of line *a* to complete the scalene triangle. Note that each line segment must be shorter than the sum of the other two segments if they are to form a triangle.

4. One method of solution is to copy each of the given angles in an adjacent manner on a straight line.

5. Using a compass radius equal to the radius of the given circle, swing arcs to mark six points on the circumference of the circle. Draw radii to each of these six points. Construct tangents to the circle at each of the six points by constructing a perpendicular line to each radius. Extend the tangents so that consecutive lines intersect to form the circumscribed hexagon.

6. Connect point *O* and point *P* and bisect *OP*. Using midpoint *N* as center and *ON* as radius, swing an arc intersecting the circle at points *R* and *S*. Connect points *R* and *P*, then points *S* and *P* to complete the required tangents.
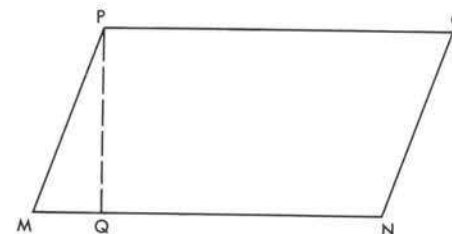
EXERCISE SET 4

1. $S = 2^{2^3} = 257$ (prime).

2. *a.* 3 is prime. *b.* 5 is prime. *c.* 17 is prime. *d.* 257 is prime. *e.* 65,537 is prime.

3. *a.* yes *b.* $\sqrt{2}$ *c.* 2 sq. units *d.* 1 sq. unit
   e. yes, since the area of *ABCD* is 1 sq. unit and the area of *AEFC* is 2×1, or 2 sq. units.

4. The volume is 64 cu. in., or eight times greater than the volume of the original cube.

EXERCISE SET 5

1.

$$a^2 + b^2 = c^2$$
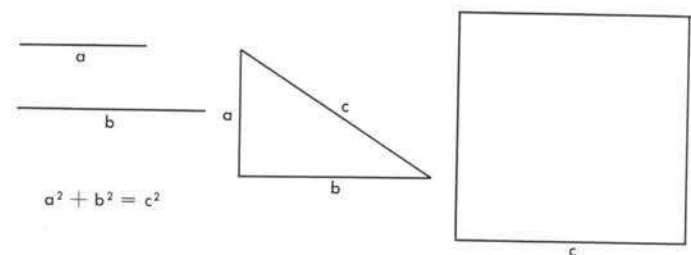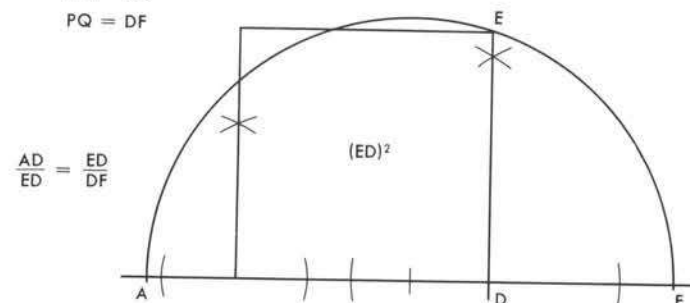
2.

$MN = AD$
$PQ = DF$

$$\frac{AD}{ED} = \frac{ED}{DF}$$

$(ED)^2 = AD \times DF$
$\therefore (ED)^2 = PQ \times MN$

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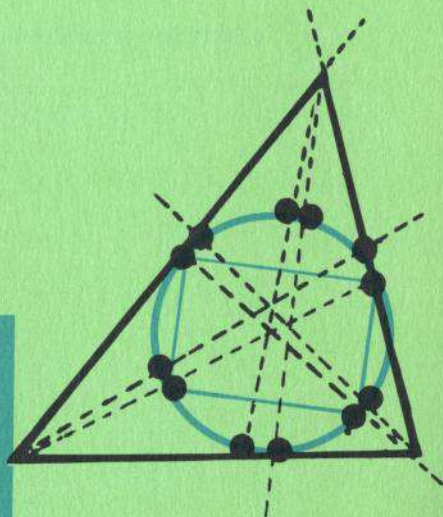
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